This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

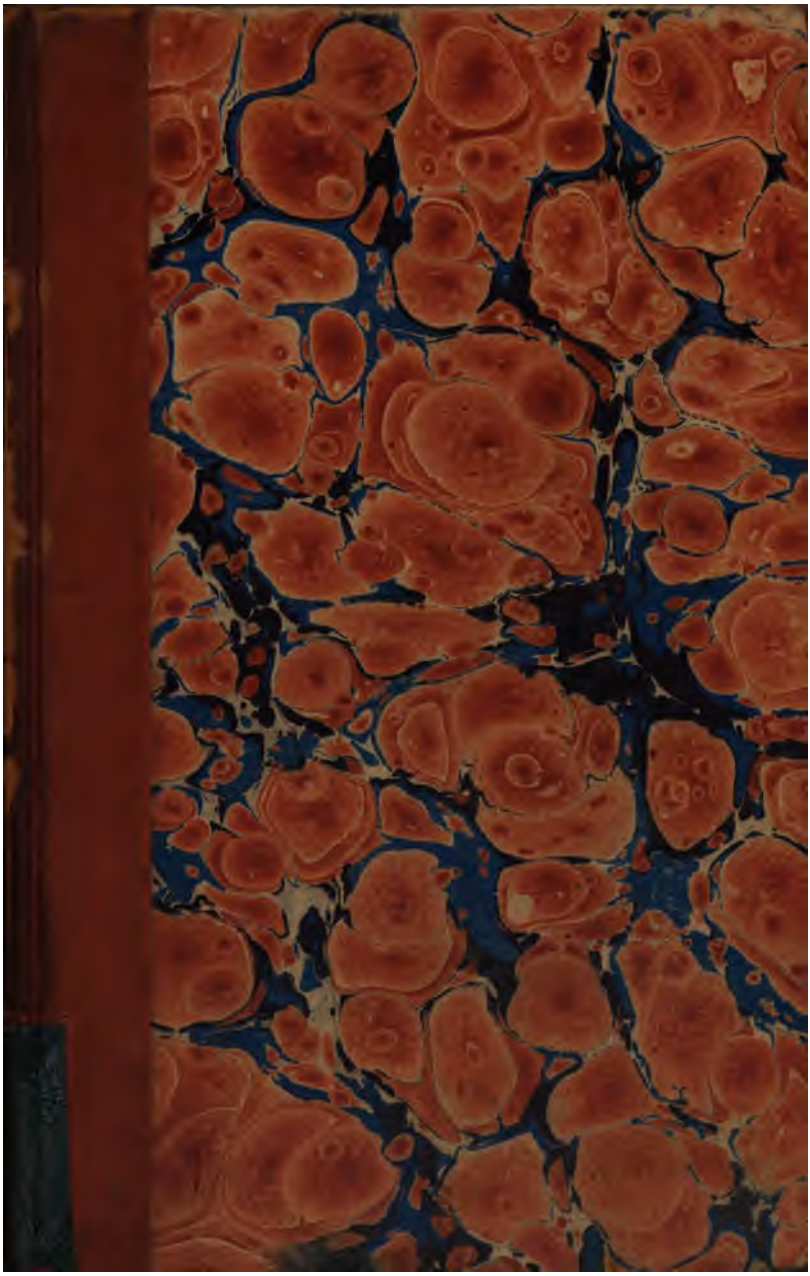
+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`



MANUAL

OF

PLANE TRIGONOMETRY.

BY

THE REV. JOSEPH A. GALBRAITH, M.A.,

FELLOW AND TUTOR OF TRINITY COLLEGE, DUBLIN,

AND

THE REV. SAMUEL HAUGHTON, M.A.,

FELLOW AND TUTOR OF TRINITY COLLEGE, DUBLIN.

SECOND EDITION.

LONDON:

LONGMAN, BROWN, GREEN, AND LONGMANS

1854.

183. c. 25

DUBLIN:
Printed at the University Press,
BY M. H. GILL.

CONTENTS.

PAGE.

INTRODUCTION, ... 1

CHAPTER I.

MEASURE OF ANGLES.

1. Angular Unit.—2. Ratio of Circumference to Diameter.—3. Division of the Circle, ... 2

CHAPTER II.

TRIGONOMETRICAL MAGNITUDES.

1. Definition of Trigonometrical Magnitudes.—2. Relations of Trigonometrical Magnitudes.—3. Complemental and Supplemental Angles.—4. Trigonometrical Tables, . . . 6

CHAPTER III.

RIGHT-ANGLED TRIANGLES.

1. Relations between the Sides and Angles.—2. Four Cases of Right Angled Triangles.—3. The Four Cases computed by Logarithmic Tables, ... 15

CHAPTER IV.

TRIGONOMETRICAL FORMULÆ, ... 25

CHAPTER V.

PAGE.

1. Relation between the Sides and Angles.—2. Expression for the Area.—3. Five Cases of Plane Triangles, ... 31

APPENDIX.

1. Definition of Logarithms.—2. Properties of Logarithms.—3. Logarithmic Tables.—4. Multiplication by Logarithms.—5. Division by Logarithms.—6. Involution by Logarithms.—7. Evolution by Logarithms.—8. Calculation of Expressions.—9. Tables of Natural Sines.—10. Tables of Logarithmic Sines, ... 51

TRIGONOMETRY.

INTRODUCTION.

TRIGONOMETRY, as the name implies (*τριγωνον μετρεω*), is that science which furnishes rules for the computation of triangles. As Geometry enables us to *construct* a triangle from three independent data, so Trigonometry enables us, from the same data expressed in numbers, to *calculate* its sides and angles. Triangles may be drawn either on a plane or spherical surface, and accordingly this branch of Mathematics is divided into *Plane* and *Spherical* Trigonometry. As a necessary preliminary to the investigation of trigonometrical rules, we proceed to explain the methods which are used for measuring angles, and expressing them by numbers.

CHAPTER I.

MEASURE OF ANGLES.

1. Angular Unit.—2. Ratio of Circumference to Diameter.—3. Division of the Circle.

1. **Angular unit.**—An angle is defined by geometers to be the inclination of one right line to another.

In order to express this kind of magnitude by numbers, it is necessary to select an *angular unit*, to which all other angles may be referred. As this selection is altogether arbitrary, different units may be proposed: in this Chapter it is not necessary to consider more than those two, which are usually employed.

The angular unit which is commonly used in mathematical treatises may be thus defined:

DEFINITION: *The angular unit* is that angle at the centre of a circle which is subtended by an arc equal in length to the radius.*

Thus if the arc AU (fig. 1) be taken equal in length to the radius CA, the angle ACU will be equal to the angular unit. Any other angle ACB may be expressed numerically, with reference to this unit, as follows:

Let N be the number which expresses its value; let a be the length of the arc AB, which subtends it; and r the radius of the circle. The following proportion is evident:

$$\angle \text{ACB} : \angle \text{ACU} :: \text{arc AB} : \text{arc AU}.$$

* For an account of the other angular unit, *vid.* p. 5.

But as ACU is the angular unit, and AU equal to the radius, it follows that

$$N : 1 :: a : r,$$

and therefore

$$N = \frac{a}{r}. \qquad (1)$$

That is, *The numerical value of an angle, is the number which expresses the ratio of the arc which subtends it, to the radius of the circle.*

In equation (1) three quantities, viz. N, a, r, are connected together, from which it follows, that if any two be given, the third may be calculated.

EXAMPLES.

1. If the radius of a circle be 35 feet, calculate the angle at the centre, which is subtended by an arc of 6 feet.

Having reduced the fraction $\frac{6}{35}$ to the decimal form, we find *Ans.* 0.17142.

2. If the radius be 12 feet 7 inches, and if the arc be 5 inches, find the angle at the centre. *Ans.* 0.03311.

3. If the radius be 97 feet, and the angle at the centre = 0.734, calculate the length of the subtending arc.

By equation (1), it is evident that 97 must be multiplied by 0 734, in order to obtain the result. *Ans.* 71.198 feet.

4. If the radius be 10 feet 9 inches, and the angle at the centre 0.01347, calculate the arc. *Ans.* 1.73763 inches.

5. An angle at the centre = 1.25 is subtended by an arc whose length is 16 feet, calculate the radius.

In this case it is evident from equation (1) that 16 must be divided by 1.25 to obtain the result. *Ans.* 12.8 feet.

6. If the angle at the centre be 0.00157, and the subtending arc 6 inches, calculate the radius. *Ans.* 318 feet, 5.65 inches.

2. **Ratio of circumference to diameter.**—The ratio of the circumference of a circle to its diameter has been determined by geometers to be as follows:

Circumference : Diameter :: 3.14159 : 1 ;

and therefore

Circumference = 3.14159 × Diameter.

As this number, 3.14159, which expresses the ratio of the circumference to the diameter, frequently occurs in mathematical formulæ, it is convenient to represent it by a single letter; that which is invariably used is the Greek letter π. Substituting this in the last equation, and $2r$ for the diameter, we have

$$\text{Circumference} = 2\pi r. \qquad (2)$$

From this equation may be derived the number which represents a right angle. The arc which subtends a right angle is equal to a fourth part of the circumference, or $\frac{1}{2}\pi r$, by equation (2). Dividing this by r, we obtain, by equation (1),

The numerical value of a right angle = $\frac{1}{2}\pi$.

The number $\frac{1}{2}\pi$, or 1.57079, which expresses the value of a right angle, referred to this particular angular unit, signifies that this unit, i. e. the angle ACU (fig. 1), is contained once, and the decimal part 0.57079 of ACU over, in a right angle. The number π, or 3.14159, which expresses two right angles, signifies that ACU is contained in two right angles, three times, and the decimal part 0.14159 of ACU over.

3. **Division of the circle.**—As the angular unit defined in section 1 is not a submultiple of four right angles, and is, moreover, too large for practical purposes, such as astronomical observations and surveying, another mode of measuring angles, and representing them numerically, has been devised, which may be thus explained :

The circumference of the circle is divided into 360 equal parts; each of these parts subtends at the centre an angle equal to the 360th part of four right angles, this angle is called a *degree*. The degree is subdivided into 60 equal parts, called *minutes*; and the minute into 60 equal parts, called *seconds*. Degrees, minutes, and seconds are represented by the symbols ° ′ ″; for example, 43° 25′ 18″, signifies 43 degrees, 25 minutes, 18 seconds.

In order to reduce an angle expressed in degrees, minutes, and seconds, to its value, referred to the angular unit, and *vice versâ*, it is necessary to find the number of seconds which are contained in this unit:

Number of seconds in angular units : number of seconds in four right angles :: arc AU : circumference.

Or,

Number of seconds in angular unit : $360 \times 60 \times 60 :: r : 2\pi r$.

From which it follows, since $\pi = 3.14159$, that

Number of seconds in angular unit $= 206265''$.

If N'' be the number of seconds in an angle which is subtended by an arc whose length is a, then

$$N'' : 206265'' :: a : \text{AU } (= r) ;$$

from which it follows that

$$N'' = 206265'' \times \frac{a}{r}. \qquad (3)$$

By equation (3), in which are connected together the radius, the number of seconds in the angle at the

centre, and the length of the arc which subtends it, any two of these quantities being given, the third may be found.

EXAMPLES.

1. In a circle of 100 feet radius, calculate the angle in degrees, minutes, and seconds, which is subtended by an arc whose length is 9 feet.

By equation (3), the number of seconds in the angle is 18563″.85, which, reduced to degrees, minutes, and seconds, gives

Ans. 5° 9′ 23″.85.

2. A person standing in the centre of a sphere observes that a line on its surface, which he knows to be 6 feet in length, subtends an angle of 3′ 28″, calculate the radius of the sphere.

Ans. 1 mile, 223.31 yds.

3. The diameter of the earth is 7926 miles, its distance from the moon is 237,638 miles, calculate the angle which the earth's diameter subtends at the moon. *Ans.* 1° 54′ 39″.6.

4. The angle which the moon subtends at the earth is observed to be 31′ 7″, calculate her diameter in miles.

Ans. 2151 miles nearly.

5. It has been ascertained that the angle which the diameter of the earth subtends at the centre of the sun is 17″.2. Calculate the sun's distance. *Ans.* 95,049,790 miles.

6. The sun's diameter subtends at the earth an angle of 32′ 3″. Calculate its diameter. *Ans.* 886,145 miles.

CHAPTER II.

TRIGONOMETRICAL MAGNITUDES.

1. Definition of Trigonometrical Magnitudes.—2. Relations of Trigonometrical Magnitudes.—3. Complemental and Supplemental Angles.—4. Trigonometrical Tables.

1. **Definition of trigonometrical magnitudes.**—There are certain magnitudes connected with angles, and entering into every trigonometrical computation, which are defined as follows:

Let the right lines (fig. 2) AA′ and DD′, be drawn intersecting at right angles in the centre of a circle *whose radius we shall suppose to be the linear unit;* let CB be drawn making an angle ACB with AC; let also BP be drawn perpendicular to AC, and AT and DS tangents to the circle at A and D respectively, and cutting the line CB produced in T and S, then

The line BP is the sine of the angle ACB.

,,	CP	,,	cosine	,,	,,
,,	AT	,,	tangent	,,	,,
,,	CT	,,	secant	,,	,,
,,	DS	,,	cotangent	,,	,,
,,	CS	,,	cosecant	,,	,,
,,	AP	,,	versed sine	,,	,,

If the numerical value of the angle ACB be denoted by A, these quantities are usually written, for brevity, thus :

$\sin A$, $\cos A$, $\tan A$, $\sec A$, $\cot A$, $\operatorname{cosec} A$, $\operatorname{versin} A$.

2. **Relations of trigonometrical magnitudes.** —The relations which these magnitudes have to each other are derived from geometrical principles as follows. Because the triangles BCP, ACT, and CDS are right-angled, it follows (Euclid, Book I. Prop. XLVII.) that

$$CB^2 = BP^2 + CP^2,$$
$$CT^2 = CA^2 + AT^2,$$
$$CS^2 = CD^2 + DS^2.$$

As the radius of the circle is supposed to be unity, $CB^2 = CA^2 = CD^2 = 1$. Substituting this, and their trigonometrical values for BP, CP, CT, &c., we have the following equations:

$$1 = \sin^2 A + \cos^2 A, \quad (1)$$
$$\sec^2 A = 1 + \tan^2 A, \quad (2)$$
$$\operatorname{cosec}^2 A = 1 + \cot^2 A. \quad (3)$$

Because the triangles CBP, CTA, and DCS are equi-angular, the following proportions follow from (Euclid, Book VI. Prop. IV.)

$$AT : AC :: BP : CP,$$
$$DS : DC :: CP : PB,$$
$$CT : CA :: CB : CP,$$
$$DS : DC :: CA : AT,$$
$$CS : CD :: CB : BP.$$

Substituting, as before, unity for the radius, and trigonometrical values for the several lines, these proportions may be written thus:

$$\tan A : 1 :: \sin A : \cos A,$$
$$\cot A : 1 :: \cos A : \sin A,$$
$$\sec A : 1 :: 1 : \cos A,$$
$$\cot A : 1 :: 1 : \tan A,$$
$$\operatorname{cosec} A : 1 :: 1 : \sin A.$$

From these proportions follow the equations

$$\tan A = \frac{\sin A}{\cos A}, \quad (4)$$

$$\cot A = \frac{\cos A}{\sin A}, \quad (5)$$

$$\sec A = \frac{1}{\cos A}, \quad (6)$$

$$\cot A = \frac{1}{\tan A}, \quad (7)$$

$$\operatorname{cosec} A = \frac{1}{\sin A}. \quad (8)$$

To these may be added,

$$\text{versin } A = 1 - \cos A, \qquad (9)$$

which follows from the fact that AP = AC − CP.

By means of the equations established in this section the value of any trigonometrical magnitude may be expressed in terms of any other.

EXAMPLES.

1. Express $\sin A$ in terms of $\cos A$.

From equation (1) it follows that $\sin^2 A = 1 - \cos^2 A$, and therefore, *Ans.* $\sin A = \sqrt{(1 - \cos^2 A)}$.

2. Express $\tan A$ in terms of $\cos A$.

By equation (4), $\tan A = \frac{\sin A}{\cos A}$, therefore,

Ans. $\tan A = \frac{\sqrt{(1 - \cos^2 A)}}{\cos A}$.

3. Express $\cos A$ in terms of $\tan A$.

From equation (6) and (2), we find

Ans. $\cos A = \frac{1}{\sqrt{(1 + \tan^2 A)}}$.

N. B.—This result might be obtained by solving the answer to the last equation for $\cos A$.

4. Express $\text{cosec } A$ in terms of $\cos A$.

Ans. $\text{cosec } A = \frac{1}{\sqrt{(1 - \cos^2 A)}}$.

5. Express $\sin A$ in terms of $\sec A$.

Ans. $\sin A = \frac{\sqrt{(\sec^2 A - 1)}}{\sec A}$.

3. **Complemental and supplemental angles.**—DEFINITIONS:—The *complement* of an angle is the difference between it and a right angle.

The *supplement* of an angle is the difference between it and two right angles.

Thus the complement of 60° is 30°; of 49° 32′, is, 40° 28′. The supplement of 60° is 120°; of 29° 37′, is 150° 23′.

The following relations exist between the sines and cosines of complemental angles :

The sine of an angle is equal to the cosine of its complement.

The cosine of an angle is equal to the sine of its complement.

Let ACB be any angle, and ACE (fig. 3) its complement, then if BP and EQ be drawn perpendicular to AC, the triangles BCP and ECQ are equal in every respect, for since QEC is the complement of ACE, it is equal to ACB; the angles CPB and CQE are equal, both being right, and BC is equal to CE, both being radii; therefore (Euc. Book I. Prop. XXVI.) BP = CQ, i. e. the sine of an angle is equal to the cosine of its complement; and CP = EQ, i. e. the cosine of an angle is equal to the sine of its complement.

The following relations exist between the sines and cosines of supplemental angles:

The sine of an angle is equal to the sine of its supplement.

The cosine of an angle is equal to the cosine of its supplement, but is of opposite sign.

Let ACB be any angle, and ACB′ (fig. 3) its supplement, then as B′CA′ is the supplement of ACB′, it is equal to ACB; from which it follows (Euclid, Book I. Prop. XXVI.) that the right-angled triangles BPC and B′P′C are equal in every respect, and therefore BP = B′P′, i. e. the sine of an angle is equal to the sine of its supplement; and CP′ = CP, i. e. the cosine of an angle is equal to the cosine of its supplement, and of opposite sign; because these lines lie, one on the right hand, the other on the left hand of the centre.

In the foregoing proof a principle is taken for granted which is assumed by mathematicians, to enable them to determine the *position*, as well as the *magnitude* of lines. It may be thus stated:—Let two right lines (fig. 4), AA′ and DD′ intersect in C, then if C be considered as the origin from which all lines are measured, every line which is measured on AA′ is positive or +, if it be situated to the *right* of C, negative or −, if to the *left*. Every line measured on DD′ is positive or +, if it lie *above* C, negative or −, if *below* C. Thus, if from M parallels be drawn to AA′ and DD′ respectively, CP is +, because it lies to the right of C; MP is + because it is measured by its equal CQ, which lies above C. Upon the same principle, CP′ and NP′ are −, MP′ is +, and NP −.

The relation between the sines and cosines of complemental angles may be expressed by the equations:

$$\sin A = \cos(\tfrac{1}{2}\pi - A), \qquad (10)$$

$$\cos A = \sin(\tfrac{1}{2}\pi - A). \qquad (11)$$

If we divide the former equation by the latter, we obtain, by equations (4) and (5)

$$\tan A = \cot(\tfrac{1}{2}\pi - A), \qquad (12)$$

If we divide the latter by the former,

$$\cot A = \tan(\tfrac{1}{2}\pi - A). \qquad (13)$$

And therefore

The tangent of an angle is equal to the cotangent of its complement.

The cotangent of an angle is equal to the tangent of its complement.

The relation between the sines and cosines of supplemental angles may be expressed by the equations:

$$\sin A = \sin(\pi - A), \qquad (14)$$

$$\cos A = -\cos(\pi - A). \qquad (15)$$

4. **Trigonometrical tables.**—It is to be remembered that the trigonometrical magnitudes defined in the first section as lines are, in reality, numbers; this follows from the supposition that the radius of the circle is equal to the linear unit. The following particular cases serve to place this in a clear point of view:

Let it be required to find the values of the sine, cosine, tangent, &c. of the angle 45°, which is equal to half a right angle; by equation (1),

$$1 = \sin^2 45° + \cos^2 45°.$$

But as the triangle BCP is in this case (fig. 5) a right-angled isosceles triangle, it is evident that $\sin 45° = \cos 45°$, and therefore,

$$1 = 2 \sin^2 45°,$$
$$1 = 2 \cos^2 45°;$$

and therefore,

$$\sin^2 45° = \cos^2 45° = \tfrac{1}{2}.$$

If the square root of $\frac{1}{2}$ or 0.5 be extracted to five places of decimals, we find that

$$\sin 45° = \cos 45° = 0.70710.$$

As the sine and cosine are equal, it follows from equations (4) and (5) that

$$\tan 45° = \cot 45° = 1.$$

It also appears from (2) and (3), and the values just found for the tangent and cotangent, that

$$\sec^2 45° = \operatorname{cosec}^2 45° = 2;$$

and therefore, extracting $\sqrt{2}$ to five places of decimals,

$$\sec 45° = \operatorname{cosec} 45° = 1.41421.$$

Let it be required to find the values of the sine, cosine, &c., of 60°. This angle is two-thirds of 90°, or of one right angle, from which it follows that the triangle ACB (fig. 6) is equilateral, and therefore that the radius AC is bisected in P; but as the radius = 1, CP = $\frac{1}{2}$; and therefore

$$\cos 60° = \tfrac{1}{2} = 0.50000.$$

From this value of the cosine we obtain that of the sine, as in Example 1. p. 9:

$$\sin 60° = \sqrt{(1 - \tfrac{1}{4})} = \sqrt{\tfrac{3}{4}};$$

extracting the square root of $\frac{3}{4} = 0.75$, to five places,

$$\sin 60° = 0.86602.$$

Let it be required to find the sine and cosine of 30°. As 30° is the complement of 60°, it follows from section 3, that

$$\sin 30° = 0.50000,$$
$$\cos 30° = 0.86602.$$

These cases are sufficient to show, that *trigonometrical magnitudes are numbers*, which are capable of being calculated from geometrical principles, and accordingly, tables, called Tables of Natural Sines, have been computed, in which the values of the sines, cosines, tangents, secants, &c., of every degree and minute in the quadrant are registered.

The statement of the method by which such tables are constructed is unsuited to the present treatise. The mode of using them in computation is explained in the Appendix.

EXAMPLES.

1. Calculate the value of the tangent of 60°.

$$\tan 60° = \frac{\sin 60°}{\cos 60°} = \frac{\sqrt{3}}{2} \div \frac{1}{2} = \sqrt{3}.$$

Ans. $\tan 60° = \sqrt{3} = 1.73205$.

2. Calculate the value of the secant of 30°.

$$\sec 30° = \frac{1}{\cos 30°} = \sqrt{\tfrac{4}{3}}.$$

Ans $\sec 30° = 1.15470$.

3. Calculate the value of the tangent of 30°.

Ans. $\tan 30° = 0.57735$.

4. Calculate the sine and cosine of 120°.

As 120° is the supplement of 60°, it follows from sect. 3, that

Ans. $\sin 120° = \sin 60° = 0.86602$,
$\cos 120° = -\cos 60° = -0.50000$.

5. Calculate the tangent of 120°.

$$\tan 120° = \frac{\sin 120°}{\cos 120°} = \frac{\sin 60°}{-\cos 60°} = -\tan 60°.$$

Ans. $\tan 120° = -1.73205$.

6. Calculate the sine of 18°.

If ACB and ACB′ (fig. 7) be each 18°, BCB′ is equal 36°, and as 36° is contained ten times in 360°, it follows that BB′ is the side of a regular decagon inscribed in the circle, and therefore that it is equal to the greater segment of the radius (= 1), cut in extreme and mean ratio (Euclid, Book IV. Prop. x.); but the greater part of unity cut in extreme and mean ratio $= \frac{\sqrt{5}-1}{2}$, and as BP, or the sine of 18°, is half BB′, it follows that

Ans. $\sin 18° = \frac{\sqrt{5}-1}{4}$
$= 0.30901$.

7. Calculate the cosine of 18°.

Ans. $\cos 18° = \frac{\sqrt{(10 + 2\sqrt{5})}}{4}$
$= 0.95105$.

8. Calculate the tangent of 18°.

Ans. $\tan 18° = \frac{\sqrt{5}-1}{\sqrt{(10 + 2\sqrt{5})}}$
$= 0.32492$.

9. Calculate the sine, cosine, and tangent of 72°.
Since 72° is the complement of 18°,

Ans. sin 72° = 0.95195,
cos 72° = 0.30901,
tan 72° = 3.07768.

CHAPTER III.

RIGHT-ANGLED TRIANGLES.

1. Relations between the Sides and Angles.—2. Four Cases of Right-Angled Triangles.—3. The Four Cases computed by Logarithmic Tables.

1. **Relations between the sides and angles.**—Let ABC (fig. 8) be a right-angled triangle, and let AN be measured off upon the hypotenuse, equal in length to the *linear unit*. With A as centre, and AN as radius, describe an arc of a circle, NM; draw NP and MT perpendicular to AC, then NP is the sine and MT the tangent of the angle A.

Since the triangles BAC and NAP are similar,

$$BC : AB :: NP : AN,$$
$$:: \sin A : 1;$$

and therefore

$$BC = AB \times \sin A.$$

As the angle B is the complement of A, it follows (Chap. II. sect. 3) that

$$BC = AB \times \cos B.$$

Similar values may be obtained for the side AC, and therefore—

PROPOSITION I.

In a right-angled triangle, either side is equal to the hypotenuse, multiplied by the sine of the opposite, or cosine of adjacent, angle.

Since the triangles BAC and TAM are similar,

$$\text{BC} : \text{AC} :: \text{TM} : \text{AM},$$
$$:: \tan A : 1 ;$$

and therefore—

$$\text{BC} = \text{AC} \times \tan A.$$

As A and B are complemental, it follows (Chap. II. sect. 3) that

$$\text{BC} = \text{AC} \times \cot B.$$

Similar values may be obtained for the side AC, and therefore—

PROPOSITION II.

In a right-angled triangle, either side is equal to the other multiplied by the tangent of the opposite, or cotangent of adjacent, angle.

In a right-angled triangle there are *five* quantities, viz.: the two sides, the hypotenuse, and the two angles adjacent to the hypotenuse. Let these be denoted by the letters a, b, c, A, B, as in (fig. 9).

The propositions established in the last section may be expressed by the equation:

$$a = c \sin \text{A} = c \cos \text{B}, \quad (1)$$
$$b = c \sin \text{B} = c \cos \text{A}; \quad (2)$$

and

$$a = b \tan \text{A} = b \cot \text{B}, \quad (3)$$
$$b = a \tan \text{B} = a \cot \text{A}. \quad (4)$$

2. **Four cases of right-angled triangles.**— If any two of the five quantities, a, b, c, A, and B, be given, the remaining three may be calculated Four cases may be proposed, viz.:

CASE I.—Given the two sides.
CASE II.—Given one side and hypotenuse.
CASE III.—Given one side and either angle.
CASE IV.—Given hypotenuse and either angle.

The following examples may be solved by means of the Tables I., II., III., in the Appendix :

CASE I.

Given a and b, it is required to find c, A, and B. By equation (3),

$$\tan A = \frac{a}{b};$$

tan A having been calculated from this equation, A may be found from the tables,

$$B = 90^\circ - A;$$

and c may be calculated from the equation,

$$c = \sqrt{(a^2 + b^2)}.$$

—(Euclid, Book I. Prop. XLVII.)

EXAMPLES.

1. Given the two sides of a right-angled triangle, equal to 500 feet and 70.27 feet respectively, find the base angles and the hypotenuse. Let $a = 70.27$ and $b = 500$; then $\tan A = 0.14054$, which corresponds in the tables to 8°; the complement of this $82^\circ = B$. If the squares of a and b be added, the sum is 254937.8729; the square root of this being taken, we find 504.91.

Ans. $c = 504.91$ feet.
$A = 8^\circ$. $B = 82^\circ$.

2. Given $a = 141.407$, and $b = 350$; find c, A, and B.

Ans. $c = 377.486$.
$A = 22^\circ$. $B = 68^\circ$.

3. Given $a = 127.38$, $b = 250$; find c, A, and B.

Ans. $c = 280.58$.
$A = 27^\circ$. $B = 63^\circ$.

CASE II.

Given a and c, it is required to find b, A, and B. From equation (1),

$$\sin A = \frac{a}{c}.$$

Sin A having been calculated from this equation, A may be found from the tables,

$$B = 90^\circ - A;$$

and b may be calculated from the equation,

$$b = \sqrt{(c^2 - a^2)}.$$

—(Euclid, Book I. Prop. XLVII.)

EXAMPLES.

1. Given $a = 5.1303$, $c = 15$; find b, A, and B.
   Ans. $b = 14.09539$.
   $A = 20^\circ$. $B = 70^\circ$.
2. Given $a = 128.76$, $c = 250$; find b, A, and B.
   Ans. $b = 214.29$.
   $A = 31^\circ$. $B = 59^\circ$.
3. Given $a = 141.57675$, $c = 175$; find b, A, and B.
   Ans. $b = 102.86$.
   $A = 54^\circ$. $B = 36^\circ$.

CASE III.

Given a and A, it is required to find B, b, and c. By equation (4),

$$b = a \cot A.$$

By equation (1),

$$c = \frac{a}{\sin A},$$

$$B = 90^\circ - A.$$

EXAMPLES.

1. Given $a = 172$, and $A = 23°$; find B, b, and c.

Ans. $B = 67°$.
$b = 405.2062$.
$c = 440.2016$.

2. Given $a = 315$, $B = 60°$; find A, b, and c.

Ans. $A = 60°$.
$b = 545.59575$.
$c = 630$.

3. Given $a = 2100$, $B = 57°$; find A, b, and c.

Ans. $A = 33°$.
$b = 3233.706$.
$c = 3855.757$.

CASE IV.

Given c and A, it is required to find a, b, and B.

By equation (1),

$$a = c \sin A.$$

By equation (2),

$$b = c \cos A,$$
$$B = 90° - A.$$

EXAMPLES.

1. Given $c = 240$, $A = 35°$; find B, a, and b.

Ans. $B = 55°$.
$a = 137.6568$.
$b = 196.596$.

2. Given $c = 575$, $B = 44°$; find A, a, and b.

Ans. $A = 46°$.
$a = 413.6205$.
$b = 399.4295$.

3. Given $c = 7$, $A = 29°$; find B, a, and b.

Ans. $B = 61°$.
$a = 3.39367$.
$b = 6.12234$.

3. **The four cases computed by logarithmic tables.**—The calculations in the examples already given are performed by means of *Tables of Natural Sines*. The examples which follow are intended as exercises in the use of *Logarithmic Tables*. The

method of performing calculations of this kind is fully explained in the Appendix.

CASE I.

Given a and b, it is required to find A, B, and c. By equation (3),

$$\tan A = \frac{a}{b}.$$

Taking the logarithm of each side,

$$\log \tan A - 10 = \log a - \log b,$$
$$\log \tan A = 10 + \log a - \log b.$$

To find c, we have by equation (1),

$$c = \frac{a}{\sin A}.$$

Taking the logarithms of both sides,

$$\log c = 10 + \log a - \log \sin A.$$

The value of A having been already found, we can, by means of this equation, find the value of $\log c$; and then, by the tables, the value of c.

EXAMPLES.

1. Given $a = 121$, and $b = 49$,

$10 + \log 121 = 12.0827854$
$\log 49 = 1.6901961$

$\log \tan A = 10.3925893$

As this number is not found in the table of log tans, we proceed by Rule XII., Appendix, in order to find the corresponding angle:

$\log \tan A = 10.3925893$
$\log \tan 67° 57' = 10.3925003$

Tab. diff. = 3631.

$890 = \text{diff.}$

$$\frac{890 \times 60}{3631} = 14''.$$

Ans. $A = 67° 57' 14''.$
$B = 22° 2' 46''.$

To find c,

$$10 + \log 121 = 12.0827854$$
$$\log \sin 97^\circ\ 57' = 9.9670125$$
$$2.1157729$$

Tab. diff. = 512. $\quad \frac{512 \times 14}{60} = 119$ (Rule XI.)

$$\log c = 2.1157610$$

By applying Rule IV. we find, *Ans.* $c = 130.545$.

N. B.—In many cases it is more expeditious to find c from the equation, $c = \sqrt{(a^2 + b^2)}$, by the rule for the extraction of the square root, than by the preceding method.

2. Given $a = 3$, and $b = 4$; find c, A, and B.

Ans. $A = 36^\circ\ 52'\ 11''$.
$B = 53^\circ\ 7'\ 49''$.
$c = 5$.

3. Given $a = 1$ mile, and $b = 3$ furlongs, 7 perches; find the remaining side and angles.

Ans. $A = 68^\circ\ 21'\ 11''$.
$B = 21^\circ\ 38'\ 49''$.
$c = 1893.54$ yds.

4. Given $a = 144$ feet, 6 inches, and $b =$ a quarter of a mile; find A, B, and c.

Ans. $A = 6^\circ\ 14'\ 50''$.
$B = 83^\circ\ 45'\ 10''$.
$c = 1327.8$ ft.

5. Given $a = 1341$, and $b = 1432$; find A, B, and c.

Ans. $A = 43^\circ\ 7'\ 13''$.
$B = 46^\circ\ 52'\ 74''$.
$c = 1961.87$.

6. Given $a = 1760$, and $b = 1000$; find A, B, and c.

Ans. $A = 60^\circ\ 23'\ 44''$.
$B = 29^\circ\ 36'\ 16''$.
$c = 2024.25$.

CASE II.

Given a and c, it is required to find b, A, and B. By equation (1),

$$\sin A = \frac{a}{c},$$

$$\log \sin A - 10 = \log a - \log c,$$
$$\log \sin A = 10 + \log a - \log c.$$

By means of this equation we can compute A. We find b by the equation,

$$b^2 = c^2 - a^2 = (c + a)\ (c - a).$$

Taking the logarithms of both sides,

$$2 \log b = \log (c + a) + \log (c - a).$$

EXAMPLES.

1. Given $a = 13.2$, and $c = 127$; find b, A, and B.

$$10 + \log 13.2 = 11.1205739$$
$$\log 127 = 2.1038037$$
$$\log \sin A = 9.0167702$$
$$\log 5^\circ\ 57' = 9.0156135$$
$$11567 = \text{diff.}$$

Tab. diff. = 12104.

$$\frac{11567 \times 60}{12104} = 57''.$$

Ans. $A = 5^\circ\ 57'\ 57''$.
$B = 84^\circ\ 2'\ 3''$.

To find b, $c + a = 140.2$; $c - a = 113.8$

$$\log 140.2 = 2.1467480$$
$$\log 113.8 = 2.0561423$$
$$2)4.2028903$$
$$\log b = 2.1014451$$

(Appendix, Rule IV.). *Ans.* $b = 126.3121$.

2. Given $a = 512$, and $c = 1007$; find the angles A and B.

Ans. $A = 30^\circ\ 33'\ 36''$.
$B = 59^\circ\ 26'\ 24''$.

3. Given $a = 32.712$, and $c = 96.2$; find the angles A and B.

Ans. $A = 19^\circ\ 52'\ 46''$.
$B = 70^\circ\ 7'\ 14''$.

4. Given $a = 123$, and $c = 157$; find A, B, and b.

Ans. $A = 51^\circ\ 34'\ 35''$.
$B = 38^\circ\ 25'\ 25''$.
$b = 97.5704$.

5. Given $a = 576$, and $c = 880$; find the angles A and B.

Ans. $A = 40° 53' 7''$.
$B = 49° 6' 53''$.

6. Given $a = 21.7$, and $c = 54.31$; find A, B, and b.

Ans. $A = 23° 33' 2''$.
$B = 66° 26' 58''$.
$b = 49.7864$.

CASE III.

Given a and A, it is required to find B, b, and c. By equation (1),

$$a = c \sin A;$$

from which,

$$\log c = 10 + \log a - \log \sin A.$$

By equation (3),

$$a = b \tan A;$$

from which,

$$\log b = 10 + \log a - \log \tan A.$$

EXAMPLES.

1. Given $a = 13$, and $A = 35° 2'$; find B, b, and c.

Ans. $B = 54° 58'$.
$b = 18.543$.
$c = 22.646$.

2. Given $a = 1157$, and $B = 58° 3' 27''$; find A, b, and c.

Ans. $A = 31° 56' 33''$.
$b = 1855.7290$.
$c = 2186.8648$.

3. Given $a = 825$, and $B = 36°$; find A, b, and c.

Ans. $A = 54°$.
$b = 599.397$.
$c = 1019.756$.

4. Given $a = 1426$, and $A = 3° 21'$; find B, b, and c.

Ans. $B = 86° 39.$
$b = 24361.38$
$c = 24403.09$

5. Given $a = 28.75$, and $A = 17° 30' 30''$; find B, b, and c.

Ans. $B = 72° 29' 30''$.
$b = 91.1371$.
$c = 95.5643$.

6. Given $a = 1000$, and $A = 18°$; find B, b, and c.

Ans. $B = 72°$.
$b = 3077.68368$.
$c = 3236.06791$.

CASE IV.

Given c and A, it is required to find a, b, and B. By equations (1) and (2),

$$a = c \sin A,$$
$$b = c \cos A;$$

from which,

$$\log a = \log c + \log \sin A - 10,$$
$$\log b = \log c + \log \cos A - 10.$$

EXAMPLES.

1. Given $c = 100$, and $A = 39° 48'$; find B, a, and b.

Ans. $B = 50° 12'$.
$a = 64.01097$.
$b = 76.82835$.

2. Given $c = 1760$, and $B = 22° 3' 56''$; find A, a, and b.

Ans. $A = 67° 56' 4''$.
$a = 1631.088$.
$b = 661.174$.

3. Given $c = 28.347$, and $A = 18° 5' 12''$; find B, a, and b.

Ans. $B = 71° 54' 48''$.
$a = 8.80047$.
$b = 26.94632$.

4. Given $c = 897.3$, and $A = 31° 21' 6''$; find B, a, and b.

Ans. $B = 58° 38' 54''$.
$a = 466.8557$.
$b = 766.2852$.

5. Given $c = 1013$, and $B = 10°$; find A, a, and b.

Ans. $A = 80°$.
$a = 997.61$.
$b = 175.90$.

6. Given $c = 50$, and $A = 6° 13' 40''$; find B, a, and b.

Ans. $B = 83° 46' 20''$.
$a = 5.424$.
$b = 49.705$.

CHAPTER IV.

TRIGONOMETRICAL FORMULÆ.

LET it be proposed to find the values of the sine and cosine of the sum of two angles, in terms of the sines and cosines of the angles themselves.

Let the angles be AOB and BOC (fig. 10); let their numerical values be A and B respectively; on OC assume a point R; let fall the perpendiculars RP and RS; from S draw SQ perpendicular, and ST parallel to OA; then, as the angles RSO and TSQ are each right, they are equal; take away TSO, which is common, and RST and QSO remain equal; the angles RTS and SQO are also equal, each being right; therefore the triangles RTS and SQO are similar, and the angle SRT equal to QOS = A.

From the right-angled triangles RTS and SOQ we obtain, by Prop. I. Chap. III.,

$$RT = RS \cos A, \quad SQ = SO \sin A;$$

and as

$$RP = RT + SQ,$$

$$RP = RS \cos A + SO \sin A.$$

If OR = R we have (Prop. I. Chap. III.)

$$RP = R \sin (A + B), \quad RS = R \sin B, \quad SO = R \cos B;$$

substituting these values, we obtain

$$R \sin (A + B) = R \sin B \cos A + R \cos B \sin A.$$

Expunging R, which is common to both sides, and changing the order of the terms,

$$\sin (A + B) = \sin A \cos B + \cos A \sin B.$$

From the same triangles we obtain

$$QO = SO \cos A, \text{ and } ST = RS \sin A;$$

and as

$$OP = QO - ST,$$

$$OP = SO \cos A - RS \sin A.$$

Substitute for OP, SO, and RS, their equals $R \cos (A + B)$, $R \cos B$, and $R \sin B$ (Prop. I. Chap. III.) we obtain

$$R \cos (A + B) = R \cos B \cos A - R \sin B \sin A;$$

therefore

$$\cos (A + B) = \cos A \cos B - \sin A \sin B.$$

We proceed to find the values of the sine and cosine of the difference of two angles in a similar manner.

Let AOB and BOC (fig. 11) be equal to A and B respectively; on OC assume a point R; let fall the perpendiculars RP and RS; from S draw SQ perpendicular, and ST parallel to OA; then, as the angles RSO and TSQ are each right, they are equal; take away RSQ, which is common, and TSR remains equal to OSQ; the angles RTS and SQO are also equal, each being right; therefore the triangles RTS and SQO are similar, and TRS = SOQ = A.

From the right-angled triangles RTS and SOQ, we obtain

$$SQ = SO \sin A, \quad RT = RS \cos A;$$

and as

$$RP = SQ - RT,$$

$$RP = SO \sin A - RS \cos A.$$

If OR = R we have (Prop. I. Chap. III.)

RP = $R \sin(A-B)$, SO = $R \cos B$, and RS = $R \sin B$;

substituting these values, we obtain

$$R \sin(A - B) = R \cos B \sin A - R \sin B \cos A;$$

therefore

$$\sin(A - B) = \sin A \cos B - \cos A \sin B.$$

From the same right-angled triangles we obtain

$$QO = SO \cos A, \quad ST = RS \sin A;$$

and as

$$OP = QO + ST,$$

$$OP = SO \cos A + RS \sin A.$$

Substituting for OP, SO, and RS, their respective values $R \cos(A-B)$, $R \cos B$, and $R \sin B$ (Prop. I. Chap. III.), we obtain

$$R \cos(A - B) = R \cos B \cos A + R \sin B \sin A;$$

therefore

$$\cos(A - B) = \cos A \cos B + \sin A \sin B.$$

The values for the sine and cosine of the sum and difference of two angles are of the highest importance, as from them are immediately derived a great variety of useful formulæ:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B, \quad (1)$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B, \quad (2)$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B, \quad (3)$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B. \quad (4)$$

By adding equations (1) and (3), we obtain

$$\sin(A + B) + \sin(A - B) = 2 \sin A \cos B.$$

By subtracting equation (3) from (1), we obtain

$$\sin(A + B) - \sin(A - B) = 2 \cos A \sin B.$$

By adding equations (2) and (4), we obtain

$$\cos(A + B) + \cos(A - B) = 2 \cos A \cos B.$$

By subtracting equation (4) from (2), we obtain

$$\cos(A + B) - \cos(A - B) = -2 \sin A \sin B.$$

These results together form the following group:

$$\sin(A + B) + \sin(A - B) = 2 \sin A \cos B, \quad (5)$$
$$\sin(A + B) - \sin(A - B) = 2 \cos A \sin B, \quad (6)$$
$$\cos(A + B) + \cos(A - B) = 2 \cos A \cos B, \quad (7)$$
$$\cos(A + B) - \cos(A - B) = -2 \sin A \sin B. \quad (8)$$

If we assume

$$A + B = A',$$
$$A - B = B',$$

it follows by addition that

$$A = \frac{A' + B'}{2};$$

by subtraction that

$$B = \frac{A' - B'}{2}.$$

Substituting these values in equation (5), we obtain:

$$\sin A' + \sin B' = 2 \sin \tfrac{1}{2}(A' + B') \cos \tfrac{1}{2}(A' - B').$$

Similar transformations may be had from substituting these values of A and B in equations (6), (7), and (8). Removing the accents from the letters A' and B' in order to preserve a uniform notation, we obtain the following group of formulæ:

$$\sin A + \sin B = 2 \sin \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A-B), \quad (9)$$
$$\sin A - \sin B = 2 \cos \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}(A-B), \quad (10)$$
$$\cos A + \cos B = 2 \cos \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A-B), \quad (11)$$
$$\cos A - \cos B = -2 \sin \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}(A-B). \quad (12)$$

As the equations (5), (6), (7), (8) are perfectly general, that is true for all values of A and B, so the equation

$$\sin A' + \sin B' = 2 \sin \tfrac{1}{2}(A'+B') \cos \tfrac{1}{2}(A'-B'),$$

derived from them is perfectly general, i. e. true for all values of A' and B', and therefore for A and B. It is for this reason that we are at liberty to remove the accents after transformation.

If in equation (1) we suppose $A = B$, we obtain

$$\sin (A+A) = \sin A \cos A + \cos A \sin A,$$

or
$$\sin 2A = 2 \sin A \cos A. \quad (13)$$

In the same way from equation (2) we obtain

$$\cos 2A = \cos^2 A - \sin^2 A.$$

If in this equation we substitute for $\cos^2 A$ its equal, $1 - \sin^2 A$, we obtain

$$\cos 2A = 1 - 2 \sin^2 A;$$

if in the same equation we substitute for $\sin^2 A$ its equal $1 - \cos^2 A$, we obtain

$$\cos 2A = 2 \cos^2 A - 1;$$

from which we have the following values for $\cos 2A$:

$$\cos 2A = \cos^2 A - \sin^2 A, \quad (14)$$
$$\cos 2A = 1 - 2 \sin^2 A, \quad (15)$$
$$\cos 2A = 2 \cos^2 A - 1. \quad (16)$$

If we divide equation (9) by equation (10), we obtain

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}{2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)},$$

or

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \tan \tfrac{1}{2}(A+B) \cot \tfrac{1}{2}(A-B).$$

Substituting for $\cot \frac{1}{2}(A-B)$ its value, the reciprocal of $\tan \frac{1}{2}(A-B)$ Chap. I. (7),

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}. \qquad (17)$$

If we divide equation (1) by (2), we obtain

$$\tan (A+B) = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}.$$

Dividing the numerator and denominator of the right-hand side by $\cos A \cos B$,

$$\tan (A+B) = \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \frac{\sin A}{\cos A} \cdot \frac{\sin B}{\cos B}}.$$

Therefore

$$\tan (A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}. \qquad (18)$$

In a similar manner from (3) and (4) we obtain

$$\tan (A-B) = \frac{\tan A - \tan B}{1 + \tan A \cdot \tan B}. \qquad (19)$$

If in (18) we make $A = B$, we obtain

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}. \qquad (20)$$

If we add together the equations

$$1 = \cos^2 A + \sin^2 A,$$
$$\cos 2A = \cos^2 A - \sin^2 A,$$

we obtain

$$1 + \cos 2A = 2\cos^2 A; \qquad (21)$$

by subtraction we obtain

$$1 - \cos 2A = 2\sin^2 A. \qquad (22)$$

If on the left-hand side of these equations we replace $2A$ by A, we must write $\frac{1}{2}A$ instead of A, on the right-hand side; therefore

$$1 + \cos A = 2\cos^2 \tfrac{1}{2}A, \qquad (23)$$

$$1 - \cos A = 2\sin^2 \tfrac{1}{2}A. \qquad (24)$$

CHAPTER V.

1. Relation between the Sides and Angles.—2. Expression for the Area.—3. Five cases of Plane Triangles.

1. **Relation between the sides and angles.**—In the triangle ABC (fig. 12), let the lengths of the sides BC, AC, and AB be denoted by a, b, and c respectively, and let the values of the opposite angles be A, B, and C. From C draw CP perpendicular to the side AB, and let its length be denoted by p; then from the right-angled triangle ACP we have (Chap. III. Prop. 1.)

$$p = b \sin A;$$

from the right-angled triangle BCP,

$$p = a \sin B;$$

therefore

$$a \sin B = b \sin A,$$
$$a : b :: \sin A : \sin B.$$

From this follows

PROPOSITION I.

The sides of a triangle are in the same ratio as the sines of the opposite angles.

This proposition remains true if one of the angles be oblique, as in fig. 13. For in the triangle BCP, $p = a \sin B$; and in the triangle ACP, $p = b \sin \text{PAC} = b \sin A$; because, since the angles A and PAC are supplemental, their sines are equal (Chap. II. Sect. 3), and therefore, as before,

$$a \sin B = b \sin A.$$

From the proportion

$$a : b :: \sin A : \sin B,$$

it follows that

$$a + b : a - b :: \sin A + \sin B : \sin A - \sin B.$$

By equation (17) Chap. IV.,

$$\sin A + \sin B : \sin B - \sin B :: \tan \tfrac{1}{2}(A + B) : \tan \tfrac{1}{2}(A - B),$$

therefore

$$a + b : a - b :: \tan \tfrac{1}{2}(A + B) : \tan \tfrac{1}{2}(A - B).$$

From this follows:

PROPOSITION II.

In a triangle the sum of the sides is to their difference, in the same ratio, as the tangent of half the sum

of the base angles, is to the tangent of half their difference.

All the relations which exist between the sides and angles of a triangle may be derived from the expression, which gives the value of the cosine of any angle in terms of the sides. This expression may be deduced as follows:—in fig. 12 let the angle A be acute, we have (Euclid, Book II. Prop. XIII.)

$$BC^2 = AC^2 + AB^2 - 2AB \times AP,$$

but $$AP = AC \cdot \cos A. \quad \text{(Chap. III. Prop. I.)}$$

Substituting this value for AP, and writing $a \cdot b$ and c, for BC, AC, and AB respectively, we obtain

$$a^2 = b^2 + c^2 - 2bc \cos A;$$

solving this equation for $\cos A$,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

Similar values may be obtained for $\cos B$ and $\cos A$, which together form the group of formulæ,

$$\left.\begin{aligned} \cos A &= \frac{b^2 + c^2 - a^2}{2bc}, \\ \cos B &= \frac{c^2 + a^2 - b^2}{2ca}, \\ \cos C &= \frac{a^2 + b^2 - c^2}{2ab}, \end{aligned}\right\} \qquad (1)$$

If the angle A be oblique, as in fig. 13, we obtain the same expression for $\cos A$, for (Euclid, Book II. Prop. XII.)

$$BC^2 = AC^2 + AB^2 + 2AB \times AP,$$

but $$AP = AC \cos PAC = -AC \cos A,$$

because PAC is the supplement of A. (Chap. II. Sect. 3.) Substituting this, we obtain, as before,

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

From the equations (1) we could obtain the values of the cosines of the angles of a triangle, and therefore of the angles themselves, if the sides were given in numbers; but as these expressions are badly adapted for calculation, we shall proceed to obtain others which offer greater facility.

If it be required to express the sine of an angle in terms of the sides, we proceed as follows:

$$\sin^2 A = 1 - \cos^2 A,$$

therefore

$$\sin^2 A = (1 + \cos A)(1 - \cos A).$$

We shall investigate the values of these factors separately, by equation (1):

$$1 + \cos A = 1 + \frac{b^2 + c^2 - a^2}{2bc},$$

$$= \frac{2bc + b^2 + c^2 - a^2}{2bc},$$

$$= \frac{(b^2 + 2bc + c^2) - a^2}{2bc},$$

$$= \frac{(b + c)^2 - a^2}{2bc}.$$

But as the difference of the squares of two quantities is equal to the product of the sum and difference,

$$1 + \cos A = \frac{(a + b + c)(b + c - a)}{2bc}; \qquad (2)$$

in the same way,

$$1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc},$$

$$= \frac{2bc - b^2 - c^2 + a^2}{2bc},$$

$$= \frac{a^2 - (b^2 - 2bc + c^2)}{2bc},$$

$$= \frac{a^2 - (b - c)^2}{2bc}.$$

Substituting, as in the last case, for the difference of the squares in the numerator the product of the difference and sum,

$$1 - \cos A = \frac{(a + c - b)(a + b - c)}{2bc}. \qquad (3)$$

Combining (2) and (3) with the equation

$$\sin^2 A = (1 + \cos A)(1 - \cos A),$$

we obtain

$$\sin^2 A = \frac{(a + b + c)(b + c - a)(a + c - b)(a + b - c)}{4b^2 c^2}. \qquad (4)$$

By the following assumptions we are enabled to throw this expression into a form which is better adapted to calculation:—If s be the *semiperimeter* of the triangle, then

$$2s = a + b + c;$$

by subtracting $2c$ from each side of this equation, we obtain

$$2(s - c) = a + b - c.$$

In a similar way,

$$2(s - b) = a + c - b,$$
$$2(s - a) = b + c - a.$$

For the factors in the numerator of equation (4), substitute these values, and we obtain

$$\sin^2 A = \frac{4s(s-a)(s-b)(s-c)}{b^2 c^2}.$$

Taking the square root of this expression for $\sin^2 A$, and of similar expressions for $\sin^2 B$ and $\sin^2 C$, we obtain

$$\left.\begin{aligned} \sin A &= \frac{2\sqrt{\{s(s-a)(s-b)(s-c)\}}}{bc}, \\ \sin B &= \frac{2\sqrt{\{s(s-a)(s-b)(s-c)\}}}{ac}, \\ \sin C &= \frac{2\sqrt{\{s(s-a)(s-b)(s-c)\}}}{ab}. \end{aligned}\right\} \quad (5)$$

If these equations be divided by a, b, and c respectively, we obtain on the right-hand side the expression

$$\frac{2\sqrt{\{s(s-a)(s-b)(s-c)\}}}{abc}.$$

This expression is said to be *symmetrical* with regard to a, b, c, because these quantities are similarly involved. From the left-hand members we obtain

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c},$$

or $$a : b : c :: \sin A : \sin B : \sin C.$$

This corresponds with the result otherwise obtained in Prop. I.

If for the value of $1 + \cos A$ obtained in equation (2) we substitute its equal, $2\cos^2 \frac{1}{2}A$ (Chap. IV. (23)); and for the factors $(a+b+c)$ and $(b+c-a)$ their values $2s$ and $2(s-a)$, we obtain

$$\cos^2 \tfrac{1}{2}A = \frac{s(s-a)}{bc}.$$

Taking the square root of this and of similar expressions for $\cos\frac{1}{2}B$ and $\cos\frac{1}{2}C$, we find

$$\left.\begin{aligned}\cos\tfrac{1}{2}A &= \sqrt{\left\{\frac{s(s-a)}{bc}\right\}},\\ \cos\tfrac{1}{2}B &= \sqrt{\left\{\frac{s(s-b)}{ac}\right\}},\\ \cos\tfrac{1}{2}C &= \sqrt{\left\{\frac{s(s-c)}{ab}\right\}}.\end{aligned}\right\} \qquad (6)$$

If we treat equation (3) in the same way, substituting $2\sin^2\frac{1}{2}A$ for $1-\cos A$ (Chap. IV. (24)), we obtain

$$\left.\begin{aligned}\sin\tfrac{1}{2}A &= \sqrt{\left\{\frac{(s-b)(s-c)}{bc}\right\}},\\ \sin\tfrac{1}{2}B &= \sqrt{\left\{\frac{(s-c)(s-a)}{ca}\right\}},\\ \sin\tfrac{1}{2}C &= \sqrt{\left\{\frac{(s-a)(s-b)}{ab}\right\}}.\end{aligned}\right\} \qquad (7)$$

If we divide the first of equations (7) by the first of equations (6), we obtain the value of $\tan\frac{1}{2}A$ (Chap. II. (4)), and similarly of $\tan\frac{1}{2}B$ and $\tan\frac{1}{2}C$, as follows:

$$\left.\begin{aligned}\tan\tfrac{1}{2}A &= \sqrt{\left\{\frac{(s-b)(s-c)}{s(s-a)}\right\}},\\ \tan\tfrac{1}{2}B &= \sqrt{\left\{\frac{(s-c)(s-a)}{s(s-b)}\right\}},\\ \tan\tfrac{1}{2}C &= \sqrt{\left\{\frac{(s-a)(s-b)}{s(s-c)}\right\}}.\end{aligned}\right\} \qquad (8)$$

2. **Expression for the area.**—In the triangle ABC, fig. 12, the area is equal to half the rectangle under the base AB and the altitude CP (Euclid, Cor. Prop. 1. Book II.); therefore, using the values already given,

$$\text{Area} = \tfrac{1}{2}cp.$$

By Prop. 1. Chap. III.,

$$p = b \sin A,$$

therefore

$$\text{Area} = \tfrac{1}{2}bc \sin A. \qquad (9)$$

From this follows

PROPOSITION III.

The area of a triangle is equal to half the product of two sides, multiplied by the sine of the included angle.

EXAMPLES.

1. Given $b = 35$ feet, $c = 117$ feet, and $A = 27°$; find the area.

$35 \times 117 = 4095$
$\sin 27° = 0.45399$

$2)1859.08905$

Area = 929.54452

Ans. Area = 929.54452 sq. ft.

2. Given $b = 28.71$ feet, $c = 103.22$ feet, and $A = 30°$; find the area. *Ans.* Area = 740.86155 sq. ft.

3. Given $b = 123.2$ feet, $c = 76$ feet, and $A = 49°$; find the area. *Ans.* Area = 3533.25033 sq. ft.

4. Given $b = 1000$ yards, $c = 2\frac{1}{2}$ miles, and $A = 42°$; calculate the area in square miles. *Ans.* Area = 0.47523 sq. miles.

Let the following examples be worked by logarithmic tables, from the equation derived from (9), as follows:

$$2 \text{ Area} = bc \sin A,$$
$$\log (2 \text{ Area}) = \log b + \log c + \log \sin A - 10.$$

5. Given $b = 876$, $c = 113$, and $A = 57° 3' 27''$; find the area.

$$\begin{aligned} \log 876 &= 2.9425041 \\ \log 113 &= 2.0530784 \\ \log \sin 57° 3' &= 9.9238373 \\ (\text{Tab. diff.} = 818) \times \tfrac{27}{60} &= \quad 368 \\ & \hline \\ & 14.9194566 \\ & 10 \\ & \hline \\ \log (2 \text{ Area}) &= 4.9194566 \\ 2 \text{ Area} &= 83072.36 \end{aligned}$$

Ans. Area = 41536.18.

6. Given $b = 513$, $c = 22$, and $A = 97°$; find the area.
As 97° is not found in the tables, we look for the sine of its supplement 83°—Chap. II. (14). *Ans.* Area = 5600.938.

7. Given $b = 127.3$, $c = 892.7$, and $A = 126°$; find the area.
Ans. Area = 45968.625.

8. Given $b = 2.314$, $c = 1.527$, and $A = 49° 6' 20''$; find the area. *Ans.* Area = 1.3355.

9. Given $b = 77$, $c = 159$, and $A = 50° 31' 28''$; find the area.
Ans. Area = 4725.15978.

10. Given $b = 287.1$, $c = 310.25$, and $A = 114° 28' 32''$; find the area. *Ans.* Area = 40534.264.

From the first equation of group (5) it follows that

$$\frac{bc \sin A}{2} = \sqrt{\{s(s-a)(s-b)(s-c)\}}.$$

From this and equation (9) it follows that

$$\text{Area} = \sqrt{\{s(s-a)(s-b)(s-c)\}}; \quad (10)$$

hence the rule for calculating the area of a triangle, which is contained in the following Proposition:

PROPOSITION IV.

1°. *Add the three sides of the triangle together, and take half the sum.*

2°. *From the half sum, subtract each side separately.*

3°. *Multiply together the half sum and the three remainders.*

4°. *Extract the square root of the product.*

The result is the area of the triangle.

EXAMPLES.

1. Given $a = 100$, $b = 86$, and $c = 72$; find the area.

$$\begin{array}{r} 100 \\ 86 \\ 72 \\ \hline 2)258 \\ \hline 129 \\ 100 \\ \hline 29 \end{array} \qquad \begin{array}{r} 129 \\ 86 \\ \hline 43 \end{array} \qquad \begin{array}{r} 129 \\ 72 \\ \hline 57 \end{array}$$

$129 \times 29 \times 43 \times 57 = 9169191$

square root $= 3028.067$

Ans. Area $= 3028.067$.

2. Given $a = 14.26$, $b = 19.2$, and $c = 32$; find the area.

Ans. Area $= 77.271$.

3. Given $a = 72$, $b = 64.1$, and $c = 22.3$; find the area.

Ans. Area $= 699.96$.

4. Given $a = 53$, $b = 49$, and $c = 98$; find the area.

Ans. Area $= 692.387$.

Let the remaining examples be worked by logarithmic tables, from the formula which is derived from equation (10), by taking the logarithm of each side.

$$\log \text{Area} = \tfrac{1}{2}\{\log s + \log (s-a) + \log (s-b) + \log (s-c)\}.$$

5. Given $a = 17.28$, $b = 13.2$, and $c = 14.62$; find the area.

$s = 22.55$, log 22.55 = 1.3531465
$s - a = 5.27$, log 5.27 = 0.7218106
$s - b = 9.35$, log 9.35 = 0.9708116
$s - c = 7.93$, log 7.93 = 0.8992732

2)3.9450419

log Area = 1.9725209

Ans. Area = 93.86871.

6. Given $a = 131$, $b = 246$, and $c = 327$; find the area.
Ans. Area = 14357.88.

7. Given $a = 2.05$, $b = 1.67$, and $c = 2.7$; find the area.
Ans. Area = 1.71012.

8. Given $a = 1800$, $b = 1728$, and $c = 1521$; find the area.
Ans. Area = 1209089.

9. Given $a = 0.23$, $b = 0.34$, and $c = 0.45$; find the area.
Ans. Area = 0.03816.

10. Given $a = 507$, $b = 603$, and $c = 721$; find the area.
Ans. Area = 150768.

3. **Five cases of plane triangles.**—In a plane triangle there are six elements, viz. the three sides, a, b, c, and the three angles, A, B, C. Any three of these quantities which are independent being given, the remaining three may be found: this readily appears from the principles of geometrical construction. The three angles, A, B, C, are not independent, since their sum is equal to 180°; and it is evident that an indefinite number of similar triangles may be constructed whose angles shall be equal respectively to three given angles, whose sum is 180°. If any other three of the six quantities, a, b, c, A, B, C, be given in numbers, the remaining three may be calculated.

The object of this section is to show the method of calculating the different cases which may be proposed, of which there are five, viz.:

CASE I.—Given one side and two angles, each adjacent to the given side.

CASE II.—Given one side and two angles, of which one is adjacent, and the other is opposite, to the given side.

CASE III.—Given two sides, and an angle opposite one of the given sides.

CASE IV.—Given two sides and included angle.

CASE V.—Given three sides.

CASE I.

Given one side and two angles, each adjacent to the given side.

Let the given side be a, and the two angles B and C; then

$$A = 180° - (B + C). \qquad (1)$$

By Prop. I. Chap. V.,

$$\frac{b}{a} = \frac{\sin B}{\sin A};$$

therefore

$$\log b - \log a = \log \sin B - \log \sin A;$$

or,
$$\log b = \log a + \log \sin B - \log \sin A. \qquad (2)$$

In a similar manner,

$$\log c = \log a + \log \sin C - \log \sin A. \qquad (3)$$

By means of the equations (2) and (3), the sides b and c may be calculated.

EXAMPLES.

1. Given $a = 217$, $B = 56° 21' 30''$, $C = 62° 41' 20''$; find A, b, and c.

By equation (1), $A = 60° 57' 10''$.

In order to find b, we proceed by equation (2), as follows:

$$\begin{aligned}
\log 217 &= 2.3364597\\
\log \sin 56^\circ\ 21' &= 9.9203519\\
(\text{Tab. diff.} = 841) \times \tfrac{10}{60} &= \quad 420\\
&\overline{12.2568536}\\
\log \sin 60^\circ\ 57' &= 9.9416090\\
&\overline{2.3152446}\\
701 \times \tfrac{10}{60} &= \quad 117\\
&\overline{}\\
\log b &= 2.3152329\\
b &= 206.64881.
\end{aligned}$$

To find c, we use equation (3):

$$\begin{aligned}
\log 217 &= 2.3364597\\
\log \sin 62^\circ\ 41' &= 9.9486495\\
(\text{Tab. diff.} = 652) \times \tfrac{10}{60} &= \quad 217\\
&\overline{12.2851309}\\
\log \sin 60^\circ\ 57' &= 9.9416090\\
&\overline{2.3435219}\\
701 \times \tfrac{10}{60} &= \quad 117\\
&\overline{}\\
\log c &= 2.3435102\\
c &= 220.55157.
\end{aligned}$$

Ans. $A = 60^\circ\ 57'\ 10''$.
$b = 206.64881$.
$c = 220.55157$.

2. Given $a = 14.83$, $B = 43^\circ\ 31'$, $C = 71^\circ\ 25'$; find A, b, and c.

Ans. $A = 65^\circ\ 4'$.
$b = 11.26096$.
$c = 15.50154$.

3. Given $a = 1728$, $B = 72^\circ\ 31'\ 30''$, $C = 81^\circ\ 24'\ 20''$; find A, b, and c.

Ans. $A = 26^\circ\ 4'\ 10''$.
$b = 3750.623$.
$c = 3887.942$.

4. Given $a = 537.21$, $B = 117^\circ\ 23'\ 12''$, $C = 52^\circ\ 18'\ 10''$; find A, b, and c.

Ans. $A = 10^\circ\ 18'\ 38''$.
$b = 2665.0558$.
$c = 2374.9082$.

5. Given $a = 1000$, $B = 120° 15' 15''$, $C = 36° 52'$; find A, b, and c.

Ans. $A = 22° 52' 45''$.
$b = 2221.768$.
$c = 1543.137$.

6. Given $a = 97.6$, $B = 36° 43' 20''$, $C = 22° 10' 15''$; find A, b, and c.

Ans. $A = 121° 6' 25''$.
$b = 68.1596$.
$c = 43.0169$.

CASE II.

Given one side and two angles, of which one is adjacent, and the other is opposite, to the given side.

This case may be immediately reduced to the preceding, for the third angle may be found by subtracting from 180° the sum of those which are given.

CASE III.

Given two sides, and an angle opposite one of the given sides.

Let the given sides be a and b, and the given angle A.

By Prop. I. Chap. V.,

$$\frac{\sin B}{\sin A} = \frac{b}{a};$$

therefore

$$\log \sin B - \log \sin A = \log b - \log a;$$

or,
$$\log \sin B = \log \sin A + \log b - \log a. \qquad (1)$$

From this equation we calculate the angle B,

$$C = 180° - (A + B); \qquad (2)$$

c may be calculated by equation (3), Case I.

This is called the *ambiguous case*, as, when the given angle is acute, it generally admits of two solutions. From equation (1) we obtain the value of log sin B, but as the sine of an angle is equal to the sine of its supplement (Chap. II. Sect. 3), it follows that to the same numerical value of log sin B corresponds two angles whose sum is 180°. This appears also from the geometrical construction of the triangle, as follows:—Let the given angle A (fig. 14) be acute, draw AC = b, from A draw a line AN of indefinite length, and making an angle CAN with AC equal to the given angle, with C as centre, and the other given side a as radius, describe a circle which will in general cut the line AN in two points, B and B′; it is evident that the two triangles ACB and ACB′ answer the given data in every respect.

The triangle BCB′ is isosceles, and therefore the angles CBB′ and CB′B are equal; and as CBB′ is the supplement of ABC, it follows that the angles ABC and AB′C, which in each of the triangles are opposite the side b, are supplemental, and therefore have the same sine.

If a be less than $b \sin A$, the case is impossible, i.e. the triangle cannot be constructed; for if CP be a perpendicular let fall on AB, CP = $b \sin A$ (Prop. 1. Chap. III.), and if a circle be described with C as centre, and a line less than CP as radius, it cannot meet the line AN.

If $a = b \sin A$, the case admits of one solution only, viz. the right-angled triangle ACP.

If a be greater than $b \sin A$, and less than b, there are two solutions, as in the above construction.

If a be greater than b, there is but one solution, because the intersection B, which in this case lies

to the left of A, must be rejected; we must select, therefore, that value of B which is acute.

If the given angle A be obtuse, it may be easily seen from following the same method of construction, that the only case which admits of a solution is that in which a is greater than b; in this case, too, we must select that value of the angle B which is acute.

EXAMPLES.

1. Given $b = 156$, $a = 130$, $A = 42° 25'$; find B, C, and a.

$$\begin{aligned} \log \sin 42° 25' &= 9.8289930 \\ \log 156 &= 2.1931246 \\ \hline & 12.0221176 \\ \log 130 &= 2.1139434 \\ \hline \log \sin B &= 9.9081742 \end{aligned}$$

By Rule XII., Appendix, we find that the value of B, which corresponds to this value of log sin B, is $54° 2' 21''$. From what has been said above, it appears that the supplement of this angle, viz. $125° 57' 39''$, has for its log sine the same value; there are, therefore, two values of B, viz.:

$$\begin{aligned} B' &= 54° 2' 21'' \\ B &= 125° 57' 39''; \end{aligned}$$

to which correspond two values of C respectively, by applying equation (2), viz.:

$$\begin{aligned} C' &= 83° 32' 39'' \\ C &= 11° 37' 21''; \end{aligned}$$

there are, of course, two values of c, which may be found by applying equation (3), Case I.; they are as follows:

$$\begin{aligned} c' &= 191.50833 \\ c &= 38.82797. \end{aligned}$$

2. Given $b = 31$, $a = 38$, and $A = 53° 20' 6''$; find B, c, and a.

$$\begin{aligned}
\log \sin 53^\circ\ 20' &= 9.9042411 \\
940 \times \tfrac{8}{60} &= \quad 94 \\
\log 31 &= 1.4913617 \\
\hline
& \quad 11.3956122 \\
\log 38 &= 1.5797836 \\
\hline
\log \sin B &= 9.8158286
\end{aligned}$$

This value of log sin B corresponds to 40° 52′ 21″. The supplement of this, 139° 7′ 39″, is inadmissible, because in this case a is greater than b. Calculating C by equation (2), and b by equation (3), Case I., we find

Ans. $B = 40^\circ\ 52'\ 21''$.
$C = 85^\circ\ 47'\ 33''$.
$c = 47.24555$.

3. Given $b = 53$, $a = 47$, $A = 36^\circ\ 42'\ 30''$; find B (acute), C, and a.

Ans. $B = 42^\circ\ 22'\ 49''$.
$C = 100^\circ\ 54'\ 41''$.
$c = 77.20757$.

4. Given $b = 312$, $a = 517$, and $A = 124^\circ\ 32'$; find B, C, and a.

Ans. $B = 29^\circ\ 48'\ 41''$.
$C = 25^\circ\ 39'\ 19''$.
$c = 271.71519$.

5. Given $b = 217$, $a = 199$, and $A = 62^\circ\ 24'\ 20''$; find B (obtuse), C, and a.

Ans. $B = 104^\circ\ 53'\ 32''$.
$C = 12^\circ\ 42'\ 8''$.
$c = 49.37320$.

6. Given $b = 30.8$, $a = 62.73$, and $A = 107^\circ\ 3'\ 13''$; find B, C, and a.

Ans. $B = 27^\circ\ 59'\ 44''$.
$C = 44^\circ\ 57'\ 3''$.
$c = 46.356$.

CASE IV.

Given two sides and included angle.

Let the given sides be a and b, and the included angle C; then

$$\tfrac{1}{2}(A + B) = 90^\circ - \tfrac{1}{2}C. \qquad (1)$$

By Prop. II. Chap. V.,

$$a + b : a - b :: \tan \tfrac{1}{2}(A + B) : \tan \tfrac{1}{2}(A - B);$$

from which,

$$\left.\begin{aligned} \log \tan \tfrac{1}{2}(A-B) = \log(a-b) + \log \tan \tfrac{1}{2}(A+B) \\ - \log(a+b). \end{aligned}\right\} (2)$$

By means of the equations (1) and (2) we can calculate the values of $\frac{1}{2}(A + B)$ and $\frac{1}{2}(A - B)$, the sum and difference of which give respectively the values of A and B. The third side c may then be calculated by equation (3), Case I.

EXAMPLES.

1. Given $a = 218$, $b = 156$, and $C = 38° 21' 20''$; find A, B, and c.

$$\begin{aligned} a + b &= 374 \\ a - b &= 62 \\ \tfrac{1}{2}(A + B) &= 70° 49' 20'' \end{aligned}$$

$$\begin{aligned} \log 62 &= 1.7923917 \\ \log \tan 70° 49' &= 10.4585322 \\ 4072 \times \tfrac{8}{60} &= 1357 \\ & \hline 12.2510596 \\ \log 374 &= 2.5728716 \\ \hline \log \tan \tfrac{1}{2}(A - B) &= 9.6781880 \end{aligned}$$

From which we find, by Rule XII. (Appendix), that

$$\tfrac{1}{2}(A - B) = 25° 29' 3''.$$

From this value of $\frac{1}{2}(A - B)$, and that already found for $\frac{1}{2}(A + B)$, we find the values of A and B.

Ans. $A = 96° 18' 23''$.
$B = 45° 20' 17''$.
$c = 136.1012$.

2. Given $a = 516$, $b = 219$, and $C = 98° 54'$; find A, B, and c.

Ans. $A = 59° 37' 18''$.
$B = 21° 28' 42''$.
$c = 590.9174$.

3. Given $a = 53.24$, $b = 31.27$, and $C = 126° 36' 6''$; find A, B, and c.

Ans. $A = 34° 8' 53''$.
$B = 19° 15' 1''$.
$c = 76.14$.

4. Given $a = 831$, $b = 536$, and $C = 16° 28' 40''$; find A, B, and c.

Ans. $A = 137° 54' 3''$.
$B = 25° 37' 17''$.
$c = 351.58$.

5. Given $a = 8214$, $b = 3732$, and $C = 61° 53'$; find A, B, and c.

Ans. $A = 91° 5' 57''$.
$B = 27° 1' 3''$.
$c = 7246$.

6. Given $a = 1.73$, $b = 1.23$, and $C = 22° 13' 30''$; find A, B, and c.

Ans. $A = 119° 34' 58''$.
$B = 38° 11' 32''$.
$c = 0.75245$.

CASE V.

Given the three sides.

If the sides be a, b, c, the values of A, B, C may be calculated by means of any of the groups of formulæ (5), (6), (7), or (8). In order to avoid the ambiguity which occurs in finding the value of an angle by means of its sine, any of the groups (6), (7), or (8), is to be preferred to (5); in using these no ambiguity can arise, because since any angle of a triangle is less than 180°, its half must be less than 90°.

From these formulæ we derive the following, which are adapted to logarithmic calculations:

$$\log \cos \tfrac{1}{2}A = 10 + \tfrac{1}{2}\{\log s + \log (s-a) - \log b - \log c\}. \quad (1)$$

$$\log \sin \tfrac{1}{2}A = 10 + \tfrac{1}{2}\{\log (s-b) + \log (s-c) - \log b - \log c\}. \quad (2)$$

$$\log \tan \tfrac{1}{2}A = 10 + \tfrac{1}{2}\{\log (s-b) + \log (s-c) - \log s - \log (s-a)\}. \quad (3)$$

EXAMPLES.

1. Given $a = 26$, $b = 31$, and $c = 43$; find A, B, and C.

$a = 26$
$b = 31$
$c = 43$

100

$s = 50$	50	50
26	31	43
$s - a = 24$	$s - b = 19$	$s - c = 7$

Using formula (1), we proceed as follows:

$\log 50 = 1.6989700$	$\log 31 = 1.4913617$
$\log 24 = 1.3802112$	$\log 43 = 1.6334685$
3.0791812	3.1248302
3.1248302	
$2)\ \bar{1}.9543510$	
$\bar{1}.9771755$	
10	

$\log \cos \frac{1}{2} A = 9.9771755$

$\frac{1}{2} A = 18° 24' 48''.46.$

In a similar manner $\frac{1}{2} B$, and $\frac{1}{2} C$ may be calculated.

Ans. $A = 36° 49' 36''.92.$
$B = 45° 36' 57''.76.$
$C = 97° 33' 25''.48.$

N. B.—The sum of these angles is 180° 0′ 0″.16, which is only 0″.16 in excess.

2. Given $a = 15.32$, $b = 21.56$, and $c = 16.22$; find the value of A. *Ans.* $A = 45° 9' 16''.$

3. Given $a = 2134$, $b = 1617$, and $c = 815$; find the value of B. *Ans.* $A = 41° 32' 31''.$

4. Given $a = 1500$, $b = 1342$, and $c = 1110$; find the value of C. *Ans.* $C = 45° 33' 35''$

5. Given $a = 1$, $b = 1.32$, and $c = 0.75$; find the value of A. *Ans.* $A = 48° 46' 24''.$

6. Given $a = 27$, $b = 32$, and $c = 9$; find the value of C. *Ans.* $C = 14° 37' 35''.$

APPENDIX.

1. Definition of Logarithms.—2. Properties of Logarithms.—3. Logarithmic Tables.—4. Multiplication by Logarithms.—5. Division by Logarithms.—6. Involution by Logarithms.—7. Evolution by Logarithms.—8. Calculation of Expressions.—9. Tables of Natural Sines.—10. Tables of Logarithmic Sines.

DEFINITION OF LOGARITHMS.

1. If the number 10 be raised to the powers, 0, 1, 2, 3, 4, &c., we obtain the series of numbers, 1, 10, 100, 1000, 10000, &c Thus:

$$10^0 = 1$$
$$10^1 = 10$$
$$10^2 = 100$$
$$10^3 = 1000$$
$$10^4 = 10000$$
&c. &c.

It is evident that for numbers intermediate to these, the powers to which 10 must be raised, so as to be equal to them, must lie between the numbers of the series, 0, 1, 2, 3, 4, &c. Thus, for all numbers lying between 100 and 1000, the corresponding power of 10 being greater than 2, and less than 3, must be 2, increased by some decimal fraction. For numbers lying between 1000 and 10000, must be 3, increased by some decimal fraction; and so on. These numbers are called *logarithms;* and in consequence of their great importance in numerical computation, their values have been calculated for all numbers from 1 to 100000, and registered in tables, called Logarithmic Tables.

DEFINITION.—*The logarithm of a number is that power to which* 10 *must be raised, so as to be equal to it. Thus:*

logarithm of 1 = 0
logarithm of 10 = 1
logarithm of 100 = 2
logarithm of 1000 = 3
&c. &c.

The logarithms of the intermediate numbers, from what has been already said, will consist of an integer and of a decimal part; the decimal part alone is registered in the tables, and is called the *mantissa;* the integer part, which is called the *characteristic*, is not entered, but may be found by

RULE I.

The characteristic of the logarithm of a number greater than unity is one less than the number of the digits of its integer part.

Thus, the characteristic of the logarithm of 849 is 2; for as this number lies between 100 and 1000, its logarithm must lie between 2 and 3. The mantissa, by referring to the tables, is found to be .9289077; therefore

$$\log 849 = 2.9289077;$$

and therefore from the definition of a logarithm,

$$849 = 10^{2.9289077}.$$

The logarithms of numbers less than unity are negative, as may be seen from the following table:

$10^{-0} = 1 = 1$
$10^{-1} = \frac{1}{10} = 0.1$
$10^{-2} = \frac{1}{100} = 0.01$
$10^{-3} = \frac{1}{1000} = 0.001$
$10^{-4} = \frac{1}{10000} = 0.0001$
&c. &c.

It follows from this, that the characteristics of the logarithms of all numbers less than unity are negative, and may be found by

RULE II.

The characteristic of the logarithm of a number less than unity, and reduced to the decimal form, is negative, and one greater than the number of ciphers following the decimal point.

A negative characteristic is denoted by writing over it the negative sign; thus, $\bar{1}$, $\bar{2}$, $\bar{3}$, &c. The reason of the rule just given may be seen from inspecting the preceding table of values of ne-

gative powers of 10, or perhaps more clearly by considering a particular case; for example, the one already proposed,

$$849 = 10^{2.9289077}.$$

If we divide each side of this equation successively by 10, which is effected on the left-hand side by removing the decimal point one place to the left; on the right-hand side, by subtracting unity from the index of the power, we obtain,

$$84.9 = 10^{1.9289077}$$
$$8.49 = 10^{0.9289077};$$

from which it follows that

$$\log 84.9 = 1.9289077$$
$$\log 8.49 = 0.9289077.$$

This corresponds with the rule given for determining the characteristic of the logarithm of a number greater than unity. If we continue the successive division of each side of the equation by 10, we obtain

$$0.849 = 10^{\bar{1}.9289077}$$
$$0.0849 = 10^{\bar{2}.9289077}$$
$$0.00849 = 10^{\bar{3}.9289077}$$
$$0.000849 = 10^{\bar{4}.9289077}$$
$$\&c. \qquad \&c.$$

From which it follows that

$$\log 0.849 = \bar{1}.9289077$$
$$\log 0.0849 = \bar{2}.9289077$$
$$\log 0.00849 = \bar{3}.9289077$$
$$\log 0.000849 = \bar{4}.9289077$$
$$\&c. \qquad \&c.$$

From this it appears, that the negative characteristics are given by the rule in question. It also appears that the logarithms of all numbers, *consisting of the same significant figures*, have the same mantissa; and that the characteristic depends solely on the position of the decimal point.

PROPERTIES OF LOGARITHMS.

2. We shall now demonstrate four propositions from which the rules for using logarithmic tables in numerical computations are derived.

PROPOSITION I.

The logarithm of the product of two numbers is equal to the sum of the logarithms of the numbers.

If the numbers be N and M, let $n = \log N$, and $m = \log M$; then, by the definition,

$$N = 10^n,$$
$$M = 10^m.$$

By multiplication,

$$N \times M = 10^{n+m};$$

therefore,

$$\log N \times M = n + m = \log N + \log M.$$

PROPOSITION II.

The logarithm of the quotient of two numbers is equal to the difference of the logarithms of the numbers.

By division,

$$\frac{N}{M} = 10^{n-m};$$

therefore

$$\log \frac{N}{M} = n - m = \log N - \log M.$$

PROPOSITION III.

The logarithm of the p^{th} power of a number is equal to p times the logarithm of the number.

If we raise each side of the equation

$$N = 10^n$$

to the power p,

$$N^p = 10^{pn};$$

therefore

$$\log N^p = pn = p \log N.$$

PROPOSITION IV.

The logarithm of the p^{th} root of a number is equal to the p^{th} part of the logarithm of the number.

If we take the p^{th} root of each side of the equation

$$N = 10^n,$$

we obtain

$$N^{\frac{1}{p}} = 10^{\frac{n}{p}};$$

therefore

$$\log N^{\frac{1}{p}} = \frac{n}{p} = \frac{\log N}{p}.$$

LOGARITHMIC TABLES

3. If the number be given, its logarithm may be found by the following rules.

If the number consist of less than six figures, we proceed by

RULE III.

Find the first four figures in the vertical column on the left, marked N; and the fifth in the horizontal column, at the top of the page. Under this, and opposite the four figures, will be found the mantissa. To this prefix the characteristic.

EXAMPLES.

1.	Find the logarithm of	35627.	*Ans.* 4.5517793.
2.	" "	56214.	*Ans.* 4.7498445.
3.	" "	832.57.	*Ans.* 2.9204208.
4.	" "	13.546.	*Ans.* 1.1318111.

If the number of digits be less than five, conceive ciphers annexed so as to make up five, and proceed as above.

5.	Find the logarithm of	127.	*Ans.* 2.1038037.
6.	" "	21.	*Ans.* 1.3222193.
7.	" "	9.	*Ans.* 0.9542425.
8.	" "	632.	*Ans.* 2.8007171.
9.	" "	5421.	*Ans.* 3.7340794.

N. B.—It may be observed that the mantissæ are arranged in sets, which have the first three figures common. In the last line of the set we sometimes find a bar printed over the first of the last four figures; this signifies that the first three figures of this and the remaining mantissæ of the line are to be taken from the next set. For example, if we look for the logarithm of 65766, we find

that the last four figures are 5014; the bar signifies that the first three figures are 818, and not 817; the mantissa is, therefore, .8180014. For the same reason, the mantissa corresponding to 65769 is 0.8180212.

10. Find the logarithm of 52845. *Ans.* 4.7230039.
11. „ „ 785.26. *Ans.* 2.8950135.
12. „ „ 25.648. *Ans.* 1.4090535.

If the number consist of more than six figures, we use

RULE IV.

1°. *Cut off the first five figures, and consider the rest as a decimal.*
2°. *Find the mantissa corresponding to the first five figures.*
3°. *Multiply the tabular difference by the decimal cut off.*
4°. *Add the integer part of this product to the last figures of the mantissa just found.*

The result is the mantissa of the required logarithm. The characteristic is found by Rules I. and II.

EXAMPLES.

1. Find the logarithm of 54673243.

Mantissa of 54673 = 0.7377729
(Tab. diff. = 80) × 0.243 = 19.44, 19
Characteristic = 7

log 54673243 = 7.7377748

2. Find the logarithm of 856324. *Ans.* 5.9326381.
3. „ „ 1725674. *Ans.* 6.2369587.
4. „ „ 32495678. *Ans.* 7.5118256.
5. „ „ 987654267. *Ans.* 8.9946049.

In the following examples we find the mantissa which corresponds to the significant figures, and prefix the negative characteristics according to Rule II.

6. Find the logarithm of 0.5634. *Ans.* $\bar{1}$.7508168.
7. „ „ 0.0876. *Ans.* $\bar{2}$.9425041.
8. „ „ 0.003564. *Ans.* $\bar{3}$.5519377.
9. „ „ 0.00085673 6. *Ans.* $\bar{4}$.9328470.

The explanation of this rule is founded on the principle, that for a small interval such as unity the increase of the number is proportional to the increase of the logarithm. Thus, if it were proposed to find the logarithm of 6732475, proceeding according to the rule, we cut off five figures, and consider what is the logarithm of 67323.75; we find by the tables that

$$\log 67323 = 4.8281635.$$

If we increase the number by unity, we find

$$\log 67324 = 4.8281699.$$

Thus for an increase of unity in the number, there is a corresponding increase of 64 in the last two figures of the logarithm; this is called the tabular difference, because it is registered in the tables. The question, then, is, if an increase of 64 in the logarithm corresponds to an increase of unity in the number, what increase of the logarithm corresponds to an increase of 0.75 in the number? This difference or increase, which is called the *proportional part*, appears from the following proportion:

$$\text{diff.} : 64 :: 0.75 : 1;$$

$$\text{diff.} = 64 \times 0.75 = 48;$$

48 is, therefore, to be added to the last two figures, in order to obtain the log 67323.75. Thus:

$$\log 67323 = 4.8281635$$
$$\text{diff. corresponding to } 0.75 = \quad 48$$
$$\log 67323.75 = 4.8281683$$
$$\log 6732375 = 6.8281683$$

If the logarithm be given, the number which corresponds to it is found by the following rules.

When the mantissa can be found in the tables, we proceed by

RULE V.

1°. *Having found the mantissa, take from the column N the four figures in the same horizontal row.*

2°. *From the head of the column take the fifth figure.*

3°. *From the characteristic, find, by the rules already given, the position of the decimal point.*

EXAMPLES.

1. Given the logarithm 2.9312137; find the corresponding number.

Having found the mantissa in the tables, the first four figures, viz. 8535, are found in the same row, in the column N; the fifth figure, 2, stands at the head of the column; therefore, all the figures together give 85352. As 2 is the characteristic, we cut off three figures for the integer part. *Ans.* 853.52.

2. Given the logarithm 3.6253433; find the number.
Ans. 4220.3.

3. Given the logarithm 4.8519062; find the number.
Ans. 71106.

4. Given the logarithm 7.3627651; find the number.
Ans. 23055000.

5. Given the logarithm 1.7253643; find the number.
Ans. 53.133.

6. Given the logarithm $\bar{2}$.6274068; find the number.
Ans. 0.042404.

In this example, having found the five figures which correspond to the mantissa, a cipher is prefixed, and then the decimal point, as the characteristic is $\bar{2}$, according to the second rule in section 1.

7. Given the logarithm $\bar{3}$.5862159; find the number.
Ans. 0.0038567.

8. Given the logarithm 0.4005725; find the number.
Ans. 2.5152.

When the mantissa cannot be found in the tables, we proceed by

RULE VI.

1°. *Having found the next lower mantissa in the tables, note the five figures which correspond, as in the former case.*

2°. *Divide the difference between this and the proposed mantissa by the tabular difference, and reduce the quotient to the form of a decimal.*

3°. *To the five figures already found add this decimal, and shift the decimal point so as to suit the characteristic of proposed logarithm.*

The result will be the required number.

EXAMPLES.

1. Given the logarithm 2.8175668; find the number.
Mantissa next lower, 8175654, to which corresponds 69700.

14 = diff.
66 = tabular diff.

$\frac{14}{66} = 0.2121$, &c.

Attaching this to the five figures, we obtain 65700.2121, &c. But as the characteristic is 2, we obtain *Ans.* 657.002121.

2. Given the logarithm 1.5673428; find the number.
Ans. 36.92689.

3. Given the logarithm 5.7326421; find the number.
Ans. 540308.875.

4. Given the logarithm 3.8924652; find the number.
Ans. 7806.6589.

5. Given the logarithm 0.9276324; find the number.
Ans. 8.4651059.

6. Given the logarithm $\bar{2}$.5321658; find the number.
Ans. 0.0340538.

7. Given the logarithm $\bar{4}$.3215467; find the number.
Ans. 0.000209675.

8. Given the logarithm $\bar{1}$.1863241; find the number.
Ans. 0.1535762.

MULTIPLICATION BY LOGARITHMS.

4. In multiplication we proceed by

RULE VII.

1°. *Find the logarithms of the numbers the product of which is required.*

2°. *Add these together, the sum will be the logarithm of the product.*

3°. *Find from the tables the corresponding number.*

This will be the required product.

The reason of this rule appears from Prop. 1. sect. 2.

EXAMPLES.

1. Find the product of 5632.1, and 32.56458.

log 5632.1 = 3.7506704
log 32.564 = 1.5127377
diff. 133 × 0.58 = 77 (Rule IV.)

5.2634158
18340 2633993 next lower in tables.

18340696 237) 165 (696 (Rule VI.)

Ans. 183406.96.

2. Find the product of 3846, and 0.056214.

log 3846 = 3.5850093
log 0.056214 = $\bar{2}$.7498445

2.3348538

Ans. 216.199.

3 Find the product of 0.3467, and 0.0098563.

log 0.03467 = $\bar{2}$.5399538
log 0.0098563 = $\bar{3}$.9937139

$\bar{4}$.5336677

Ans. 0.0003417178.

4. Find the product of 1576.3, and 0.85673.

Ans. 1350.463.

5. Find the product of 18.54 and 735.6425.

Ans. 13638.811.

6. Find the product of 21.357 and 6324.567.

Ans. 135073.76.

7. Find the product of 18.21, 35.672, 2.847, and 11.256.

Ans. 20816.56.

8. Find the product of 0.0873, 25.773206, and 0.005693.

Ans. 0.0128092.

DIVISION BY LOGARITHMS.

5. In division we proceed by the following—

RULE VIII.

1°. *Find the logarithms of the numbers the quotient of which is required.*

2°. *Subtract the logarithm of the divisor from that of the dividend; the difference will be the logarithm of the quotient.*

3°. *Find from the tables the corresponding number.*

This will be the required quotient.

The reason of this rule appears from Prop. II. sect. 2.

EXAMPLES.

1. Find the quotient of 876.32 by 32.567.

log 876.32 = 2.9426627
log 32.567 = 1.5127778

1.4298849
14 26908

162) 35 (216 The additional figs.

We thus obtain for the figures of the quotient, 26908216; and as the characteristic is 1, *Ans.* 26.908216.

2. Find the quotient of 0.32567 by 0.0129. *Ans.* 25.24573.
3. Find the quotient of 567.342 by 110.673. *Ans.* 5.12629.
4. Find the quotient of 0.01237 by 108.46. *Ans.* 0.00011405.
5. Find the quotient of 385.27 by 10056789. *Ans.* 0.000038309.
6. Find the quotient of 287.642 by 83456. *Ans.* 0.00344663.

INVOLUTION BY LOGARITHMS.

6. In involution we proceed by the following—

RULE IX.

1°. *Find the logarithm of the number which is to be raised to the given power.*

2°. *Multiply this logarithm by the index of the given power; the product will be the logarithm of the required power.*

3°. *Find from the tables the corresponding number.*

This will be the required power.

The reason of this rule appears from Prop. III. sect 2.

EXAMPLES,

1. Find the fourth power of 8.7.

log 8.7 = 0.9395193
4

3.7580772
12 57289

76) 60 (789 Additional figures.

The figures of the power are 5728978947; and as the characteristic is 3, *Ans.* 5728.978.

2. Find the fifth power of 11. *Ans.* 161051.
3. Find the square of 82.56. *Ans.* 6816.153.
4. Find the cube of 196.3. *Ans.* 7564163.
5. Find the twenty-first power of 1.5. *Ans.* 4987.9.
6. Find the 297th power of 1.037. *Ans.* 48562.
7. Find the third power of 0.03564.

log 0.03564 = $\bar{2}$.5519377
3

5.6558131

The characteristic $\bar{5} = 3 \times \bar{2} + 1$ (carried).

Ans. 0.00004527027.

8. Find the square of 0.08567. *Ans.* 0.00733935.
9. Find the tenth power of 0.10341.

Ans. 0.00000000001398379.

EVOLUTION BY LOGARITHMS.

7. In evolution we proceed by the following—

RULE X.

1°. *Find the logarithm of the number whose root is to be found.*
2°. *Divide this logarithm by the index of the given root; the quotient will be the logarithm of the required root.*
3°. *Find from the tables the corresponding number.*

This will be the required root.

The reason of this rule appears from Prop. IV. sect. 2.

EXAMPLES.

1. Find the fifth root of 56732.

log 56732 = 5) 4.7538281

0.9507656
39 89282

49) 17 (347 Additional figures.

The figures are, therefore, 89282347; and as the characteristic is 0,

Ans. 8.9282347.

2. Find the square root of 7586.321. *Ans.* 87.10948.
3. Find the cube root of 1729. *Ans.* 12.00231.
4. Find the twenty-eighth root of 100000. *Ans.* 1.50859.
5. Find the thirty-fifth root of 1.0653. *Ans.* 1.001809.
6. Find the third root of 0.0003256.

$$\log 0.0003256 = 3)\ \bar{4}.5126844$$

$$\bar{2}.8375614$$

Ans. 0.06887957.

N. B.—In cases like the preceding, in which the characteristic is negative, and is not divisible by the given index, it is necessary to *increase the negative characteristic by as many units as will render it divisible, and then carry so much to the decimal.* Thus, in the preceding example we consider

$$\bar{4} = \bar{6} + 2\text{; and therefore}$$

$$3)\ \bar{6} + 2.5126844$$

$$\bar{2} + 0.8375614.$$

7. Find the fifth root of 0.0096532.

$$\log 0.0096532 = \bar{3}.9846713\text{;}$$

$$5)\ \bar{5} + 2.9846713$$

$$\bar{1} + 0.5969342$$

Ans. 0.3953067.

8. Find the square root of 0.1478. *Ans.* 0.3844476.
9. Find the eighteenth root of 0.01789. *Ans.* 0.7996920.
10. Find the 100th root of 1000000. *Ans.* 1.1481537.

CALCULATION OF EXPRESSIONS.

8. It appears from the examples already given that arithmetical operations are greatly abridged by the use of Logarithmic Tables. In the calculation of complex expressions they may be said to be absolutely essential.

The expressions to which this kind of calculation is immediately applicable are those, the parts of which are connected by the signs of multiplication, division, involution, and evolution; if otherwise, they must by transformation be made to assume such a form.

EXAMPLES.

1. Calculate the value of $\frac{1856 \times 31^5 \times 7.81^{10}}{612^2 \times 1928^3}$.

		log 1856 =	3.2685780
2 log 612 =	5.5735028	5 log 31 =	7.4568085
3 log 1928 =	9.8553210	10 log 7.81 =	8.9265100
	15.4288238		19.6518965
			15.4288238
			4.2230727
		Ans.	16713.70.

2. Calculate the value of $\frac{71 \times (5.6)^3 \times (89)^{\frac{1}{2}}}{(3.57)^7 \times (39)^{\frac{1}{2}}}$.

		log 71 =	1.8512583
7 log 3.57 =	3.8686774	3 log 5.6 =	2.2445640
$\frac{1}{2}$ log 39 =	0.5303548	$\frac{1}{2}$ log 89 =	0.9746950
	4.3990322		5.0705173
			4.3990322
			0.6714851
		Ans.	4.6933731.

3. Calculate the value of $\frac{8^5 \times 21^2 \times 56^3}{48^3 \times 112^4}$.

Ans. 0.1458334.

4. Calculate the value of $\frac{(113)^{\frac{1}{2}} \times (8563)^{\frac{1}{3}} \times (562)^{\frac{1}{4}}}{(98732)^{\frac{1}{7}} \times (3462)^{\frac{1}{4}}}$.

Ans. 31.4799.

TABLES OF NATURAL SINES.

9. The nature of Tables of Natural Sines and Cosines, &c., has been fully explained, Chap. II. sect. 4. In using these tables we have either to find the sine, cosine, tangent, &c., of an angle whose value is given in degrees, minutes, and seconds; or, being given the value of the angle in degrees, minutes, and seconds, to find the corresponding value of the sine, cosine, &c.

If the value of the angle be given in degrees and minutes, the sine, cosine, &c., is found directly from the tables, in which are

registered the values of trigonometrical quantities corresponding to every degree and minute in the quadrant.

If the angle contains seconds, we must proceed by the *method of proportional parts*, as in the following examples.

RULE XI.

1°. *Find from the tables the sine, cosine, &c., which corresponds to the degrees and minutes.*

2°. *Multiply the tabular difference by the seconds, and divide by* 60.

3°. *If the required quantity be a sine, tangent, or secant, add the result to the last figures obtained in* 1°; *if it be a cosine, cotangent, or cosecant, subtract.*

The result will be the required sine, cosine, &c.

EXAMPLES.

1. Find the sine of 6° 36′ 27″.

$$\sin 6^\circ\ 36' = 0.1149372$$
$$\text{Tab. diff.} = \frac{2889 \times 27}{60} = 1300.$$
$$1300$$
$$0.1150672$$

Ans. $\sin 6^\circ\ 36'\ 27'' = 0.1150672$.

2. Find the cosine of 31° 28′ 42″.

$$\cos 31^\circ\ 28' = 0.8529440$$
$$\text{Tab. diff.} = \frac{1519 \times 42}{60} = 1063.$$
$$1063$$
$$0.8528377$$

Ans. $\cos 31^\circ\ 28'\ 42'' = 0.8528377$.

The reason of this rule is founded on a principle similar to that from which Rule II. is derived, viz. that for a small interval, such as one minute, the increase of the sine is proportional to the increase of the angle.

3. Find the tangent of 39° 21′ 46″. *Ans.* 0.8203218.
4. Find the cotangent of 76° 53′ 8″. *Ans.* 0.2329731.
5. Find the sine of 86° 3′ 17″. *Ans.* 0.9976302.
6. Find the cosine of 57° 32′ 36″. *Ans.* 0.5366616.

If the value of the sine, cosine, &c., be given, and it is required to find the angle, we use the following rule—

RULE XII.

1°. *Find in the tables the next lower sine, cosine, &c., and note the corresponding degrees and minutes.*

2°. *Subtract this from the given sine, cosine, &c., multiply the difference by 60, divide by the tabular difference, and consider the result as seconds.*

3°. *If the given value be that of a sine, tangent, or secant, add these seconds to the degrees and minutes found in 1°; if it be that of a cosine, cotangent, or cosecant, subtract.*

The result will be the required angle.

EXAMPLES.

1. Given the sine = 0.7321564; find the angle.

$$\begin{array}{rr} & 0.7321564 \\ \sin 47^\circ\ 4' = & 0.7321467 \\ \hline & 97 \end{array}$$

Tab. diff. = 1982. $\quad \frac{97 \times 60}{1982} = 3''.$

Ans. 47° 4′ 3″.

2. Given the cosine = 0.8532672; find the angle.

$$\begin{array}{rr} & 0.8532672 \\ \cos 31^\circ\ 26' = & 0.8532475 \\ \hline & 197 \end{array}$$

Tab. diff. = 1517. $\quad \frac{197 \times 60}{1517} = 7''.$

Ans. 31° 25′ 53″.

3. Given the tangent = 1.5632417; find the angle.

Ans. 57° 23′ 35″.

4. Given the cotangent = 0.4763215; find the angle.

Ans. 64° 31′ 50″.

5. Given the secant = 1.8345672; find the angle.

Ans. 56° 58′ 9″.

6. Given the cosecant = 2.1346521; find the angle.

Ans. 27° 56′ 4″.

TABLES OF LOGARITHMIC SINES.

10. In order to apply logarithmic calculations to trigonometrical quantities, it is necessary to construct tables of the logarithms of the natural sines, cosines, &c. As all the sines and cosines, all the tangents from 0° to 45°, and all the cotangents from 45° to 90°, are less than unity, the logarithms of these quantities have negative characteristics. In order to avoid the necessity of entering negative numbers, 10 *is added to every logarithm* before it is registered in the tables of logarithmic sines. Thus, on referring to the table of natural sines, we find that

$$\sin 16° = 0.2756374.$$

If we calculate the logarithm of 0.2756374, we find that its value is $\bar{1}.4403381$; if to this we add 10, we find that

$$\log \sin 16° = 9.4403381.$$

As in the use of tables of natural sines two questions present themselves:—1°. Being given the angle in degrees, minutes, and seconds, required the log sine, log cosine, &c. 2°. Being given the log sine, log cosine, &c., required the value of the angle in degrees, minutes, and seconds.

If the value of the angle be given in degrees, minutes, and seconds, we proceed by

RULE XIII.

1°. *Find from the tables the log sine, log cosine, &c., which corresponds to the degrees and minutes,*

2°. *Multiply the tabular difference by the seconds, and divide by* 60.

3°. *If the required quantity be a log sine, log tangent, or log secant, add the result to the last figures obtained in* 1°; *if it be a log cosine, log cotangent, or log cosecant, subtract.*

The result will be the required log sine, log cosine, &c.

EXAMPLES.

1. Find the log sine of 6° 36′ 27″.

$$\log \sin 6° 36' = 9.0604604$$
$$4907$$
$$9.0609511$$

$$\frac{(\text{Tab. diff.} = 10905) \times 27''}{60} = 4907.$$

Ans. $\log \sin 6° 36' 27'' = 9.0609511.$

2. Find the log cosine of 13° 5′ 32″.

log cos 13° 5′ = 9.9885776
157

9.9885619

$$\frac{\text{(Tab. diff.} = 294) \times 32}{60} = 156.8.$$

Ans. log cos 13° 5′ 32″ = 9.9885619.

3. Find the log tangent of 42° 47′ 26″.

Ans. log tan 42° 47′ 26″ = 9.9664721.

4. Find the log cotangent of 73° 21′ 7″.

Ans. log cot 73° 21′ 7″ = 9.4757096.

If the value of the log sine, log cosine, &c., be given, and it is required to find the angle, we use the following—

RULE XIV.

1°. *Find in the tables the next lower log sine, log cosine, &c., and note the corresponding degrees and minutes.*

2°. *Subtract this from the given log sine, log cosine, &c., multiply the difference by* 60, *divide by the tabular difference, and consider the results as seconds.*

3°. *If the given value be that of a log sine, log tangent, or log secant, add these seconds to the degree and minutes found in* 1°; *if it be that of a log cosine, log cotangent, or log cosecant, subtract.*

The result will be the required angle.

EXAMPLES.

1. Given log sine = 9.3562413; find the angle.

9.3562413
log sin 13° 7′. 9.3559007

3406

Tab. diff. = 5419.

$$\frac{3406 \times 60}{5419} = 37''.$$

Ans. 13° 7′ 37″.

2. Given log cotangent = 10.1123463; find the angle.

10 1123463
log cot 37° 41′. 10.1121446

2017

Tab. diff. = 2612

$$\frac{2017 \times 60}{2612} = 46''.$$

Ans. 37° 40′ 14″.

3. Given log cosine = 9.8732415; find the angle.

Ans. 41° 40′ 50″.

4. Given log tangent = 9.7963423; find the angle.

Ans. 32° 1′ 58″.

From the manner in which these tables are constructed, it is evident that previously to using log sines, log cosines, &c., in numerical computations, it is necessary to subtract 10 from the values entered in the tables. The mode of operation is illustrated in the following examples.

EXAMPLES.

1. Find the value of the angle A from the equation,

$$\sin A = \frac{113}{355}.$$

Take the logs of each side; we obtain

$$\log \sin A - 10 = \log 113 - \log 355;$$

$$\log \sin A = 10 + \log 113 - \log 355$$

$$= 9.5028500.$$

Ans. $A = 18° 33′ 38″$.

2. Find the value of the angle A from the equation,

$$\sin^2 A = \frac{287}{1184}.$$

$$2 (\log \sin A - 10) = \log 287 - \log 1184;$$

$$2 \log \sin A = 20 + \log 287 - \log 1184;$$

$$2 \log \sin A = 19.3845302;$$

$$\log \sin A = 9.6922651.$$

Ans. $A = 29° 29′ 40″$.

TABLE I.

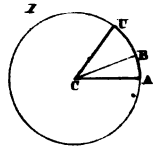
	SINE.		TANGENT.	
0°	.00000		.00000	90°
1°	.01745		.01745	89°
2°	.03490		.03492	88°
3°	.05233		.05241	87°
4°	.06975		.06992	86°
5°	.08715		.08749	85°
6°	.10453		.10510	84°
7°	.12187		.12278	83°
8°	.13917		.14054	82°
9°	.15643		.15838	81°
10°	.17365		.17632	80°
11°	.19081		.19438	79°
12°	.20791		.21255	78°
13°	.22495		.23087	77°
14°	.24192		.24933	76°
15°	.25882		.26795	75°
16°	.27563		.28674	74°
17°	.29237		.30573	73°
18°	.30901		.32492	72°
19°	.32557		.34432	71°
20°	.34202		.36397	70°
21°	.35837		.38386	69°
22°	.37460		.40402	68°
23°	.39073		.42447	67°
24°	.40673		.44523	66°
25°	.42262		.46631	65°
26°	.43837		.48773	64°
27°	.45399		.50952	63°
28°	.46947		.53171	62°
29°	.48481		.55431	61°
30°	.50000		.57735	60°
	COSINE.		COTANGENT.	

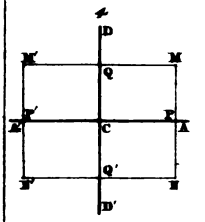
TABLE II.

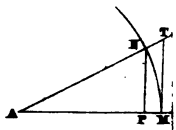
	SINE.		TANGENT.	
30°	.50000		.57735	60°
31°	.51504		.60086	59°
32°	.52992		.62487	58°
33°	.54464		.64941	57°
34°	.55919		.67451	56°
35°	.57357		.70021	55°
36°	.58778		.72654	54°
37°	.60181		.75355	53°
38°	.61566		.78128	52°
39°	.62932		.80978	51°
40°	.64279		.83910	50°
41°	.65606		.86928	49°
42°	.66913		.90040	48°
43°	.68200		.93251	47°
44°	.69466		.96569	46°
45°	.70710		1.00000	45°
46°	.71934		1.03553	44°
47°	.73135		1.07237	43°
48°	.74314		1.11061	42°
49°	.75471		1.15037	41°
50°	.76604		1.19175	40°
51°	.77714		1.23489	39°
52°	.78801		1.27994	38°
53°	.79863		1.32704	37°
54°	.80901		1.37638	36°
55°	.81915		1.42815	35°
56°	.82904		1.48256	34°
57°	.83867		1.53986	33°
58°	.84805		1.60033	32°
59°	.85716		1.66428	31°
60°	.86602		1.73205	30°
	COSINE.		COTANGENT.	

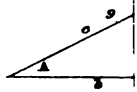
TABLE III.

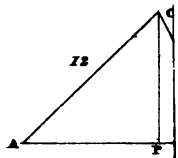
	SINE.		TANGENT.	
60°	.86602		1.73205	30°
61°	.87462		1.80405	29°
62°	.88295		1.88072	28°
63°	.89100		1.96261	27°
64°	.89879		2.05030	26°
65°	.90631		2.14450	25°
66°	.91354		2.24603	24°
67°	.92050		2.35585	23°
68°	.92718		2.47508	22°
69°	.93358		2.60509	21°
70°	.93969		2.74747	20°
71°	.94552		2.90421	19°
72°	.95105		3.07768	18°
73°	.95630		3.27085	17°
74°	.96126		3.48741	16°
75°	.96592		3.73205	15°
76°	.97029		4.01078	14°
77°	.97437		4.33147	13°
78°	.97815		4.70463	12°
79°	.98162		5.14455	11°
80°	.98481		5.67128	10°
81°	.98769		6.31375	9°
82°	.99027		7.11537	8°
83°	.99254		8.14434	7°
84°	.99452		9.51436	6°
85°	.99619		11.43005	5°
86°	.99756		14.30066	4°
87°	.99863		19.08113	3°
88°	.99939		28.63625	2°
89°	.99985		57.28996	1°
90°	1.00000		Infinite.	0°
	COSINE.		COTANGENT.	

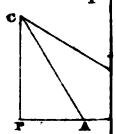

1
T
B
C
A


4
D
M'
M
Q
F'
F
E'
C
A
Q'
E
D'


T
M
F
M


9
o
A
B


12
A
F


C
F
A